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par

Alain ROUET

RENORMALISATION DES THEORIES A SYMETRIE BRISEE

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PRESENTATION

CHAPITRE I : Equivalence Theorems for Effective Lagrangians.

CHAPITRE II : Ward Identities and Trace Identities for the Stress-Energy Tensor.

CHAPITRE III: Noether's Theorem and Ward Identities in Renormalizable Theories.

CHAPITRE IV : Renormalizable Models with Broken Symmetries.

CHAPITRE V : Renormalization of the Abelian Higgs-Kibble Model.

PRESENTATION

Cette thèse comprend une série d'articles publiés, la plupart en collaboration, ces deux dernières années. Ces articles portent sur l'étude des brisures de symétrie en théorie perturbative des champs.

Le formalisme utilisé est celui de la théorie de la renormalisation de N.N. Bogoliubov dans la version donnée par W. Zimmermann.

Le premier article est consacré à la preuve de deux théorèmes combinatoires ayant trait à ce formalisme ; l'un a trait au problème des valeurs moyennes dans le vide non nulles engendrées par la brisure de symétrie ; l'autre a trait aux problèmes de symétrie brisée lorsqu'aucune masse nulle n'intervient dans la théorie symétrique.

Suivent deux lettres écrites en collaboration avec R. Stora : la première concerne le tenseur impulsion-énergie ; y sont démontrées l'identité de Ward, l'identité d'asymétrie et l'identité de trace anormale ; la seconde définit les identités de Ward renormalisées pour les théories à symétrie brisée.

L'article suivant, écrit en collaboration avec J.H. Lowenstein, R. Stora et W. Zimmermann est un cours donné à l'école d'été internationale de Capri (juillet 1973) sur "Renormalisation et invariance en théorie quantique des champs". On y trouve un développement détaillé des problèmes abordés dans les deux lettres ; par ailleurs cet article traite la resommation des brisures superrenormalisables lorsque la théorie symétrique n'a pas de masse nulle, suivant une méthode due à M. Gomes, J.H. Lowenstein, W. Zimmermann, et présente une première approche des théories de jauge.

Un exemple de ces théories est traité en grand détail dans le dernier article, écrit en collaboration avec C. Becchi et R. Stora. Vu la longueur de cet article, je renvoie à son introduction pour l'exposé de son contenu détaillé.

C H A P I T R E I

EQUIVALENCE THEOREMS FOR EFFECTIVE LAGRANGIANS

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EQUIVALENCE THEOREMS FOR EFFECTIVE LAGRANGIANS

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Abstract: Two combinatorial theorems relevant to Zimmermann's formulation of Lagrangian perturbation theory are proved in view of application to problems concerning symmetry breaking.

1. Introduction

An efficient version of Lagrangian renormalizable theories has been given by Zimmermann [1] and followers [2–5] for which a number of useful combinatorial theorems have been proved. Some of these theorems require however some slight extension when super-renormalizable couplings are present. Many practical examples are most easily formulated in a way that requires such couplings, for instance those in which symmetry-breaking phenomena are to be described [3, 6]. It turns out however that even in such cases it is possible and useful to give a description in terms of an effective Lagrangian involving only strictly renormalizable couplings [3, 6]. It is the goal of the present paper to establish two equivalence theorems which allow such a formulation.

Sect. 2 is devoted to the statement and proof of a "translation" theorem which allows to treat as a parameter of the theory the field vacuum expectation value. Sect. 3 is devoted to the statement and proof of an equivalence theorem between a set of renormalizable, possibly super-renormalizable couplings, with an other set of strictly renormalizable couplings. Sect. 4 is devoted to a couple of applications: the φ^3 theory and the description of some general features of symmetry-breaking phenomena.

In all that follows, unless otherwise stated, we shall deal with theories involving one neutral scalar field. The corresponding Green functions $\langle T\varphi(x_1), \ldots, \varphi(x_n)\rangle$ will be denoted $\langle X_n\rangle$.

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2. A translation theorem

Although this theorem is a particular case of an equivalence theorem due to Lam [4], we include it here for the reader's convenience, in view of its relevance to problems involving broken symmetries.

Let us consider an effective Lagrangian [1]

$$\mathscr{L}^0_{\text{eff}}(x) = (\tfrac{1}{2}(\partial_\mu\varphi)^2 + \sum_\alpha K_\alpha N_{\delta(\alpha)}\mathscr{L}_\alpha)(x), \tag{1}$$

where the K_α are constant or space-time dependent coupling constants, the $\mathscr{L}_\alpha$ are Wick monomials of the scalar field φ and its derivatives, namely formal expressions of the type $\Pi_i(D_i\varphi)\varphi^a$, where the symbols D_i indicate some differentiation with respect to x; the symbol $N_{\delta(\alpha)}$ indicates that, in the perturbation theory associated with $\mathscr{L}^0_{\text{eff}}$, power counting is applied by prescribing power counting index $\delta(\alpha)$ to vertices associated with $\mathscr{L}_\alpha$, which requires $\delta(\alpha)$ to be larger than or equal to the canonical dimension of $\mathscr{L}_\alpha$, obtained by adding one unit per scalar field or derivative of the first order. Let us now define

$$\mathscr{L}^F_{\text{eff}} = \tfrac{1}{2}(\partial_\mu\varphi)^2 + \sum_\alpha K_\alpha \mathscr{L}^F_\alpha, \tag{2}$$

with the following definition for the $\mathscr{L}^F_\alpha$:

$$\mathscr{L}^F_\alpha = \sum_{p=0}^{a}\binom{a}{p}F^p N_{\delta(\alpha)-p}\Big(\prod_{i\in\mathscr{M}}^{a} D_i\varphi\Big)\varphi^{a-p}. \tag{4}$$

We shall express the connected Green functions computed from $\mathscr{L}^F_{\text{eff}}$, namely $\langle X_n\rangle^c_F$, in terms of those corresponding to $F=0$.

One has *theorem 1*:

$$\langle x_n\rangle^c_F = \langle x_n\rangle^c_{F=0}, \qquad \text{if} \quad n>1,$$
$$\langle\varphi(x_1)\rangle^c_F = \langle\varphi(x_1)\rangle^c_{F=0} - F,$$

with the corollary

$$\langle\mathscr{L}^F_\alpha(x), X_n\rangle^c_F = \langle\mathscr{L}_\alpha(x), X_n\rangle^c_{F=0}.$$

The content of theorem 1 and its corollary asserts that, performing a translation of the interacting field by amount F is equivalent to perform the same translation within the effective Lagrangian according to the precise description for the power counting summarized in eq. (4).

The proof of the corollary follows from theorem 1 by differentiation with respect to K_α [2].

The proof of the theorem relies on the integrated effective equation of motion [1, 2]:

$$\langle i\int d^4x\left[\frac{\delta\mathscr{L}^F}{\delta\varphi}\right](x), X_n\rangle^c_F = 0 \qquad \text{if} \quad n>1,$$
$$= -1 \quad \text{if} \quad n=1, \tag{5}$$

where $[\delta\mathscr{L}^F/\delta\varphi]'(x)$ is obtained from $[\delta\mathscr{L}^F/\delta\varphi](x)$ by dropping the constant terms.

On the other hand, using Lowenstein insertion formula [2], we get

$$\frac{\partial}{\partial F}\langle X_n\rangle_F^c = \langle i\int d^4x\left[\frac{\partial\mathscr{L}^F}{\partial F}\right]'(x),\, X_n\rangle_F^c \tag{6}$$

where $[\partial\mathscr{L}^F/\partial F]'(x)$ is obtained from $\partial\mathscr{L}^F/\partial F(x)$ by dropping the constant terms. Taking into account the formula

$$\frac{\delta}{\delta\varphi}N_\delta\Big(\prod_{i\in\mathscr{A}}D_i\varphi\Big)(\varphi+F)^p = p\sum_{q=0}^{p-1}\binom{p-1}{q}F^q N_{\delta-q-1}\Big(\prod_{i\in\mathscr{A}}D_i\varphi\Big)\varphi^{p-q-1}$$

$$= \frac{\partial}{\partial F}N_\delta\Big(\prod_{i\in\mathscr{A}}D_i\varphi\Big)(\varphi+F)^p\,, \tag{7}$$

one finds

$$\frac{\partial}{\partial F}\langle X_n\rangle_F^c = \langle i\int d^4x\left[\frac{\partial\mathscr{L}^F}{\partial F}\right](x),\, X_n\rangle_F^c$$

$$= \langle i\int d^4x\left[\frac{\delta\mathscr{L}^F}{\delta\varphi}\right]'(x),\, X_n\rangle_F^c\,. \tag{8}$$

Thus

$$\frac{\partial}{\partial F}\langle X_n\rangle_F^c = 0 \qquad \text{if} \quad n>1,$$

$$\frac{\partial}{\partial F}\langle\varphi(x)\rangle = -1. \tag{9}$$

Hence

$$\langle X_n\rangle_F^c = \langle X_n\rangle_{F=0}^c \qquad \text{if} \quad n>1,$$

$$\langle\varphi(x)\rangle_F^c = \langle\varphi(x)\rangle_{F=0}^c - F. \tag{10}$$

3. An equivalence theorem for super-renormalizable couplings*

We shall prove the following lemma for

$$\mathscr{L}_{\text{eff}} = \mathscr{L}(\varphi)+\alpha N_p\varphi^p+a_1\varphi+a_2N_4\varphi^2+a_3N_4\varphi^3+a_4N_4\varphi^4+a_5N_4(\partial_\mu\varphi)^2, \tag{11}$$

where $\mathscr{L}(\varphi)$ is an arbitrary renormalizable Lagrangian. It is possible to find the a_i, $1\leqslant i\leqslant 5$, as formal power series in α and the parameters of $\mathscr{L}(\varphi)$, so that the Green

* After completion of this work I learnt from Professor W. Zimmermann that he independently established theorems of the type of those contained in sect. 3 of this paper.

functions defined by $\mathscr{L}_{\text{eff}}$ do not depend on α. (The Green functions involving normal product operators [1] do however, in general, depend on α.)

The proof goes as follows.

Lowenstein's insertion formula [2] yields

$$\frac{\partial}{\partial\alpha}\langle X_n\rangle^c = \langle \widetilde{N_p\varphi^p}(0), X_n\rangle^c + \sum_{k=1}^{4} \frac{\partial a_k}{\partial\alpha}\langle \widetilde{N_4\varphi^k}(0), X_n\rangle^c$$

$$+ \frac{\partial a_5}{\partial\alpha}\langle \widetilde{N_4(\partial_\mu\varphi)^2}(0), X_n\rangle^c . \tag{12}$$

A slight generalization of Zimmermann's identity [1], [2] which allows to take into account the presence of super-renormalizable couplings yields

$$\langle \widetilde{N_p\varphi^p}(0), X_n\rangle^c = \sum_{k=1}^{4} r_k\langle \widetilde{N_4\varphi^k}(0), X_n\rangle^c + r_5\langle {}_4\widetilde{(\partial_\mu\varphi)^2}(0), X_n\rangle^c , \tag{13}$$

where the r_i are formal power series in α, $a_i (1 \leqslant i \leqslant 5)$, and the parameters of $\mathscr{L}(\varphi)$, which can be computed by restricting n to values smaller or equal to four and evaluating the corresponding Green functions at zero momentum. In the absence of superrenormalizable couplings, the simplicity of the normalization conditions [1] for $N_\delta\varphi^\delta$ yields $r_1 = 0$ and $r_p = 1$.

Putting together eq. (12) and eq. (13) one gets

$$\frac{\partial}{\partial\alpha}\langle X_n\rangle^c = \sum_{k=1}^{4}\left(r_k + \frac{\partial a_k}{\partial\alpha}\right)\langle \widetilde{N_4\varphi^k}(0), X_n\rangle^c$$

$$+\left(r_5 + \frac{\partial a_5}{\partial\alpha}\right)\langle N_4\widetilde{(\partial_\mu\varphi)^2}(0), X_n\rangle^c . \tag{14}$$

According to the result of Gomes and Lowenstein [5], the independence of $\langle X_n\rangle^c$ with respect to α is equivalent to the set of conditions

$$r_i + \frac{\partial a_i}{\partial\alpha} = 0, \qquad 1 \leqslant i \leqslant 5, \tag{15}$$

which can be solved for the a_i with the boundary conditions

$$a_i(\alpha = 0) = 0, \qquad 1 \leqslant i \leqslant 5. \tag{16}$$

The variation of the Green functions involving a normal product operator $N_\delta\mathscr{M}(x)$ [1], is accordingly controlled by:

$$\frac{\partial}{\partial\alpha}\langle N_\delta\mathscr{M}(x), X_n\rangle^c = \langle N_\delta\mathscr{M}(x), \widetilde{N_p\varphi^p}(0), X_n\rangle + \sum_{k=1}^{4}\frac{\partial a_k}{\partial\alpha}\langle N_\delta\mathscr{M}(x), \widetilde{N_4\varphi^k}(0), X_n\rangle^c$$

$$+ \frac{\partial a_5}{\partial\alpha}\langle N_\delta\mathscr{M}(x), N_4\widetilde{(\partial_\mu\varphi)^2}(0), X_n\rangle^c . \tag{17}$$

Since, however, Zimmermann's identity now contains additional terms [2, 3]

$$\langle N_\delta \mathscr{M}(x), \widetilde{N_p \varphi^p}(0), X_n\rangle^c = \sum_{k=1}^{4} r_k \langle N_\delta \mathscr{M}(x), \widetilde{N_4 \varphi^k}(0), X_n\rangle^c$$

$$+ r_5 \langle N_\delta \mathscr{M}(x), N_4 \widetilde{(\partial_\mu \varphi)^2}(0), X_n\rangle^c + \sum_{\substack{k \leqslant \delta \\ i}} s_k^i \langle N_\delta Q_k^i(x), X_n\rangle^c, \qquad n \neq 0, \tag{18}$$

where the Q_k^i form an independent set of operators of dimension k, one is left with

$$\frac{\partial}{\partial \alpha} \langle N_\delta \mathscr{M}(x), X_n\rangle^c = \sum_{\substack{k \leqslant \delta \\ i}} s_k^i \langle N_\delta Q_k^i(x), X_n\rangle^c . \tag{19}$$

It is now easy to find coefficients $\mathscr{M}_k^i(\alpha)$ such that the Green functions

$$\langle N_\delta \mathscr{M}(x), X_n\rangle^c = \sum_{\substack{k \leqslant \delta \\ i}} \mathscr{M}_k^i(\alpha) \langle N_\delta Q_k^i(x), X_n\rangle^c \tag{20}$$

are independent of α:

$$\frac{\partial}{\partial \alpha} \sum_{\substack{k \leqslant \delta \\ i}} \mathscr{M}_k^i(\alpha) \langle N_\delta Q_k^i(x), X_n\rangle^c = \sum_{\substack{k, k' \leqslant \delta \\ i, i'}} \left[\frac{\partial \mathscr{M}_{k'}^{i'}(\alpha)}{\partial \alpha} \delta_{ii'} \delta_{kk'} + \mathscr{M}_{k'}^{i'}(\alpha) s_{k'k}^{i'i} \right]$$

$$\times \langle N_\delta Q_k^i(x), X_n\rangle^c , \tag{21}$$

where $s_{k'k}^{i'i}$ is the relevant analogue of s_k^i in eq. (18).

The left-hand side of eq. (21) vanishes provided that

$$\frac{\partial \mathscr{M}_k^i}{\partial \alpha}(\alpha) + \sum_{\substack{k' \leqslant \delta \\ i'}} \mathscr{M}_{k'}^{i'}(\alpha) s_{k'k}^{i'i} = 0. \tag{22}$$

This differential system can be solved with boundary conditions

$$\mathscr{M}_k^i(\alpha = 0) = M_k^i , \tag{23}$$

where M_k^i is a numerical matrix which characterizes invariantly a local operator of maximum dimension δ.

On now has *theorem 2*: Let $\mathscr{L}_{\min}(\varphi)$ be an effective renormalizable Lagrangian which involves super-renormalizable couplings L_i with coefficient a_i, and $\langle X_n\rangle(a_i)$ the corresponding Green functions considered as formal power series in the a_i's. There exists an effective Lagrangian $\mathscr{L}_4(\varphi)$ where all terms are assigned dimension four, which yields the same Green functions. (Under such a change of Lagrangian, Green functions involving normal product operators *do not* remain unchanged.)

Let us first use the "translation" theorem to cast $\mathscr{L}_{\min}(\varphi)$ into the form

$$\mathscr{L}'_{\min}(\varphi) = L_4 + a_1 \varphi + a_2 N_2 \varphi^2 + a_3 N_3 \varphi^3 , \tag{24}$$

where L_4 is a set of terms of dimension four and a_1 is determined by the condition $\langle \varphi \rangle = 0$.

Now apply the lemma to $\mathscr{L}'_{\min}(\varphi) - A_2 N_2 \varphi^2 + \Lambda_4(A_2, \varphi)$ where the set of dimension four couplings $\Lambda_4(A_2, \varphi)$ is so adjusted that the Green functions stay invariant, and then choose $A_2 = a_2$.

The Lagrangian now assumes the form

$$\mathscr{L}''_{\min}(\varphi) = L'_4 + a'_1 \varphi + a_3 N_3 \varphi^3 \tag{25}$$

Now adjust $\Lambda'_4(A_3, \varphi)$ so that $\mathscr{L}''_{\min}(\varphi) - A_3 N_3 \varphi^3 + \Lambda'_4(A_3, \varphi)$ yields the same Green functions, and choose $A_3 = a_3$. The final Lagrangian contains only dimension four couplings and no more linear term since the condition $\langle \varphi \rangle = 0$ remains fulfilled.

4. Two applications

4.1. The φ^3 theory

Let

$$\mathscr{L} = \tfrac{1}{2} N_4 [(\partial_\mu \varphi)(\partial^\mu \varphi) - m^2 \varphi^2] - a N_2 \varphi^2 - g N_3 \varphi^3 - c\varphi \,, \tag{26}$$

where c, a, g are adjusted in such a way that the following normalization conditions hold:

$$\begin{aligned} &\langle \varphi \rangle = 0, \\ &\Gamma_2(m^2) = 0, \\ &\Gamma_3(m^3, m^3, m^2) = \lambda, \end{aligned} \tag{27}$$

where Γ_2 and Γ_3 are the two and three points vertex functions respectively.

One could add into $\mathscr{L}$ a term of the form $\frac{1}{2} b(\partial_\mu \varphi)(\partial^\mu \varphi)$ and impose the extra normalization condition

$$\Gamma'_2(m^2) = i. \tag{28}$$

The same Green functions can be obtained from the dimension four Lagrangian

$$\mathscr{L}_4 = \tfrac{1}{2}(1+\beta)\partial_\mu \varphi \partial^\mu \varphi - \tfrac{1}{2}(m^2 + \alpha)\varphi^2 - \gamma \varphi^3 - \delta \varphi^4 \,, \tag{29}$$

where $\alpha, \beta, \gamma, \delta$ are formal power series in λ restricted by the requirement that the source of the φ field has dimension two. The effective equation of motion [1, 2] is (for $n > 1$)

$$\langle [(1+\beta)\Box \varphi + (m^2 + \alpha)\varphi + 3\gamma N_3 \varphi^3 + 4\delta N_3 \varphi^3](x), X_n \rangle^c = 0. \tag{30}$$

Zimmermann's identity [1, 2] is of the form

$$\begin{aligned} \langle N_3 \varphi^2(x), X_n \rangle^c = {} & \langle N_2 \varphi^2(x), X_n \rangle^c + r \langle N_3 \varphi^3(x), X_n \rangle^c \\ & + s \langle \Box \varphi(x), X_n \rangle^c . \end{aligned} \tag{31}$$

Hence the constraint

$$4\delta + 3\gamma r = 0, \tag{32}$$

which together with the normalization conditions eq. (27), eq. (28) allows to recursively define α, β, γ, δ.

*4.2. Symmetry breaking**

We consider a multiplet of scalar field φ transforming under a finite dimensional representation $\mathscr{D}$ of a compact Lie group G, and an effective Lagrangian

$$\mathscr{L}_{\text{eff}} = \mathscr{L}_{\text{sym}} + B_1 \cdot \varphi + B_2 \cdot \varphi \otimes \varphi + B_3 \cdot \varphi \otimes \varphi \otimes \varphi$$
$$\sim \mathscr{L}_{\text{sym}} + \mathscr{L}_{\text{Break}}, \tag{33}$$

where $\mathscr{L}_{\text{sym}}$ is invariant under G and the B_i are some tensor valued coupling constants which characterize the symmetry breaking.

We assume explicitly that $\mathscr{L}_{\text{sym}}$ contains a non-vanishing mass term of dimension four so that we may consider the Green functions as formal power series in the B_i together with a set of other parameters, collectively denoted Λ, which characterize $\mathscr{L}_{\text{sym}}$, and the parameter $\hbar$, which counts the number of loops in Feynmann diagrams as is recalled in appendix A.

We know from the translation and the equivalence theorem that there exists an effective Lagrangian $\mathscr{L}_4$ of dimension four, without a linear term, which yields the Green functions for

$$\varphi - \langle \varphi \rangle = \varphi - F. \tag{34}$$

The coefficients in $\mathscr{L}_4$ are expressible in B_2, B_3, Λ, $\hbar$, B_1 being eliminated through the condition $\langle \varphi \rangle = F$. Let $M^2(B_2, B_3, F, \Lambda)$ be the zeroth order term in $\hbar$ of the mass term in $\mathscr{L}_4$.

If one expresses $\mathscr{L}_4$ in terms of M^2, F, $\hbar$ and the necessary complementary set of parameters, the corresponding set of Green functions coincide with those computed from $\mathscr{L}_{\text{eff}}$, when expanded in terms of the B_i, $\hbar$, Λ through the substitution

$$M^2 \to M^2(B_2, B_3, F, \Lambda),$$
$$F \to F(B_i, \hbar, \Lambda). \tag{35}$$

Whereas the foregoing argument shows the existence of a dimension four Lagrangian which yields a partially summed up version of the Green functions computed form $\mathscr{L}_{\text{eff}}$, it is more efficient to determine $\mathscr{L}_4$ *via* the characterisation of $\mathscr{L}_{\text{eff}}$ *via* Ward identities [6]. Application of the effective Noether theorem [2, 7] to $\mathscr{L}_{\text{eff}}$ yields Ward identities of the form

$$\partial_\mu \langle j^N_\mu(x), X_n \rangle^c = \sum_{i=1}^{n} \vec{\theta}^i \delta(x - x_i) \langle X_n \rangle^c + \left\langle \frac{\delta \mathscr{L}_{\text{Break}}}{\delta \vec{\omega}}(x), X_n \right\rangle^c, \tag{36}$$

* Refs. [3, 6].

where $\vec{J}_\mu^N$ is the Noether current associated with $\mathscr{L}_{sym}$, counted with dimension three, $\delta\mathscr{L}_{Break}/\delta\vec{\omega}$ is the variation of $\mathscr{L}_{Break}$ induced by an infinitesimal variation of $\boldsymbol{\varphi}$ under the action of G, and $\vec{\theta}^i$ is the infinitesimal generator of G in representation $\mathscr{D}$ acting on $\boldsymbol{\varphi}^i$.

Under the change of Lagrangian $\mathscr{L}_{eff} \to \mathscr{L}_4$ we know that the formal expressions for $\vec{J}_\mu^N$ and $\delta\mathscr{L}_{Break}/\delta\vec{\omega}$ in terms of normal ordered products change in such a way that the maximum dimension is preserved together with the covariance under exact symmetries (Lorentz invariance, invariance under the residual internal symmetry group, i.e. the little group of $\mathscr{L}_{Break}$ in G).

$\vec{J}_\mu^N$ *does not* transform into the Noether current for $\mathscr{L}_4$ and the known information (maximum dimension) on $\delta\mathscr{L}_{Break}/\delta\vec{\omega}$ does not take into account the full covariance properties of $\mathscr{L}_{Break}$. The correct form of $\vec{J}_\mu$ can however directly be computed once the precise form of $\delta\mathscr{L}_{Break}/\delta\vec{\omega}$ is known [2, 3, 5, 7, 8]. The latter can only be recovered through a detailed analysis of some features of the high momentum behavior of the two and three points functions.

As a last remark we should like to point out that our working hypothesis according to which $\mathscr{L}_{sym}$ contains a non vanishing mass term is highly restrictive. Many interesting models do not fulfill this condition, e.g. σ models involving fermion fields [6, 8], models involving gauge fields. In these cases the $\mathscr{L}_4$ Lagrangian has to be directly determined by Ward identities which are suitable generalization of the expression of Noether's theorem for the classical Lagrangian generating the tree approximation of the theory at hand.

I wish to thank R. Stora and E. Tirapequi for constant help during the course of this work. I also wish to thank J. H. Lowenstein, B. Schroer and W. Zimmermann for constantly keeping me informed about their work prior to publication.

Appendix. Counting the number of loops in a Feynman diagram

The following is a quick summary of some results contained in ref. [9]. Let

$$\langle X(J)\rangle = \langle \exp \frac{i}{\hbar}\int \mathscr{L}(\varphi)+J\varphi\rangle \tag{A.1}$$

be the generating functional for the Green functions. In terms of $\bar{\varphi} = \varphi/\sqrt{\hbar}$ one can write

$$\begin{aligned}\langle X(J)\rangle &= \langle \exp \frac{i}{\hbar}\int \mathscr{L}(\sqrt{\hbar}\bar{\varphi})+J\sqrt{\hbar}\bar{\varphi}\rangle \\ &= \langle \exp i\int \frac{1}{\hbar}\mathscr{L}(\sqrt{\hbar}\bar{\varphi})+\frac{J}{\sqrt{\hbar}}\bar{\varphi}\rangle.\end{aligned} \tag{A.2}$$

We see that a term of degree k in $\bar{\varphi}$ comes, in the action, together with a coefficient $\hbar^{\frac{1}{2}(k-2)}$. Thus, in a Feynman graph, a vertex with k lines carries a coefficient proportional to $\hbar^{\frac{1}{2}(k-2)}$, whereas the Feynman propagator does not depend on $\hbar$.

Besides the n-point function carries an extra factor $\hbar^n$. Now it can easily be seen that for a graph with N_e external lines, $\mathcal{N}$ loops, N_i internal vertices with i lines, one has the relation

$$N_e = 2-N_1+N_3+2N_4-2\mathcal{N}. \tag{A.3}$$

According to the above remark the corresponding power of $\sqrt{\hbar}$ for the corresponding Green function is

$$n = -N_1+N_3+2N_4+2N_e. \tag{A.4}$$

Hence

$$\mathcal{N} = 1+\tfrac{1}{2}n-\tfrac{1}{2}N_e, \tag{A.5}$$

which shows that the power of $\hbar$ counts the number of loops.

As an example (see fig. 1):

$N_e = 5$, $\mathcal{N} = 3$,

$N_1 = N_2 = 0$, $N_3 = 1$,

$N_4 = 4$, $n = 9$.

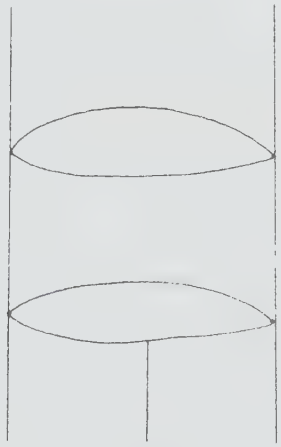

Fig. 1

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C H A P I T R E II

WARD IDENTITITES AND TRACE IDENTITIES FOR THE STRESS-ENERGY TENSOR

LETTERE AL NUOVO CIMENTO VOL. 4, N. 4 27 Maggio 1972

Ward Identities and Trace Identities for the Stress-Energy Tensor.

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(ricevuto il 18 Febbraio 1972)

Recent progress in the theory of perturbative renormalization [1,2] have allowed to give an accurate description of local operators in the perturbative sense.

In particular, for some simple renormalizable interaction Lagrangians, it has been shown that one can construct stress-energy tensor operators whose Green's function fulfill Ward identities [3,4]. These Ward identities make up one of the building blocks of the work by CALLAN, COLEMAN, JACKIW [5] whose program can now be carried out rigorously whereby the phenomena connected with broken scale invariance which were later pointer out by CALLAN [6], SYMANZIK [7], can be recovered. We now have at our disposal two methods which allow to deal efficiently with the construction of the renormalized perturbation expansion. They exhibit the common feature that they do not involve any ultraviolet regularization procedure. The first method, due to EPSTEIN and GLASER [1] is of a recursive nature. The second one, due to ZIMMERMANN [2] exhibits the answer explicitly.

Whereas both methods were found suitable to show the possibility of fulfilling Ward identities, the so-called trace identity [5] which, as it is well known, involves anomalies, could only be obtained by Zimmermann's method. Since the calculations involved are lengthly, they will not be given here, but will rather be deferred to a later publication.

For the sake of definiteness, we shall here adhere to the notations of ZIMMERMANN [2] and LOWENSTEIN [3]. We shall denote by $\mathscr{L}_{ett}$ an effective renormalizable Lagrangian

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(1) H. EPSTEIN and V. GLASER: CERN preprint TH 1400.

(2) W. ZIMMERMANN: *Lectures on elementary particles and quantum field theory*, in *Brandeis University, Summer Institute in Theoretical Physics*, Vol. 1 (Cambridge, 1970).

(3) J. H. LOWENSTIEN: Pittsburg preprint N.Y.O. 3829-67. Most formulae of interest concerning equations of motion and reduction of normal operator products can be found in this article. However, our philosophy concerning Ward identities is different.

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in the sense of ZIMMERMANN, where the free-field part is counted with dimension four ([5]), and the interaction part with whatever dimension is used to construct renormalized Feynman graphs for the Green's functions of the fields.

Keeping these specifications in mind, mere applications of Noether's theorem to $\mathscr{L}_{\text{eff}}$ ([3]), and eventually of Belinfante's trick ([6]), produce a stress-energy tensor operator $\Theta_{\mu\nu}$ whose Green's functions fulfill the following Ward identities:

$$(1) \qquad \partial_\mu \langle \Omega, T\Theta^{\mu\nu}(\xi)\varphi_1(x_1)\dots\varphi_n(x_n), \Omega\rangle = \sum_{k=1}^{n} \frac{1}{i}\delta(\xi - x_k)\,\partial_k^\nu \langle \Omega, T\varphi_1(x_1)\dots\varphi_n(x_n), \Omega\rangle ,$$

and the asymmetry identity

$$(2) \qquad \langle \Omega, T\Theta^{\mu\nu}(\xi)\varphi_1(x_1)\dots\varphi_n(x_n), \Omega\rangle - \langle \Omega, T\Theta^{\nu\mu}(\xi)\varphi_1(x_1)\dots\varphi_n(x_n), \Omega\rangle =$$

$$= \sum_{k=1}^{n} \delta(\xi - x_k)\, S_k^{\mu\nu} \langle \Omega, T\varphi_1(x_1)\dots\varphi_n(x_n), \Omega\rangle ,$$

where $\{S_k^{\mu\nu}\}$ denotes the set of the relativistic spin matrices appropriate to field $\{\varphi_k\}$. Equations (1) and (2) imply the expected Ward identity for the relativistic angular-momentum density. One can in fact show by straight-forward computation that there exists a family of stress energy tensor operators of dimension four which fulfill the above identities, the allowed ambiguity being a linear combination of terms of the form $(\Box g_{\mu\nu} - \partial_\mu\partial_\nu)\varphi_k^2$ extended to all possible boson fields.

For each of such operator one can derive a trace identity of the form ([5])

$$(3) \qquad g_{\mu\nu}\langle \Omega, T\Theta^{\mu\nu}(\xi)\varphi_1(x_1)\dots\varphi_n(x_n), \Omega\rangle =$$

$$= \langle \Omega, T\theta(\xi)\varphi_1(x_1)\dots\varphi_n(x_n), \Omega\rangle + \frac{1}{i}\sum_{k=1}^{n} d_k\,\delta(\xi - x_k)\langle \Omega, T\varphi_1(x_1)\dots\varphi_n(x_n), \Omega\rangle .$$

In this equation $\theta(\xi)$ denotes a linear combination of the terms surviving in the effective interaction Lagrangian after use has been made of the effective equations of motion.

The coefficient of these terms, as well as the dimension coefficients d_k occuring in eq. (3) are computable formal power series in the coupling constants. All of them are linear in the coefficient of the terms $(\Box g_{\mu\nu} - \partial_\mu\partial_\nu)\varphi_k^2$ in $\theta_{\mu\nu}$. In particular, it is easy to see that no choice of this parameter can be made so that $\theta(\xi)$ may become « soft ». Suitable combinations of the trace identity, (eq. (3)), and the Ward identity, (eq. (1)), then yield Ward identities for the currents associated with dilation and conformal transformations.

Let now k be the momentum variable conjugate to ξ under Fourier transformation. It is possible to eliminate the Green's functions of $\Theta_{\mu\nu}$ at $k = 0$ between the trace identity, (eq. (3)), and the Ward identity, (eq. (1)), once differentiated with respect to k at $k = 0$. This yields a family of relations between the Green's functions of the fields and their derivatives with respect to the coupling constants which stem from the

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([5]) R. JOST: *The general theory of quantized fields.*

([6]) B. SCHROER: Pittsburg preprint NYO 3992-77. This paper displays calculations similar to those leading to our eq. (3).

Green's functions of $\theta(\xi)$ computed at $k = 0$. One thus obtains a family of identities whose manifold is associated with the ambiguities in the definition of the stress-energy tensor operator.

These equations can presumably be cast into the form found by LOWENSTEIN ([10]) who identifies them as the Callan-Symanzik and renormalization group equations.

* * *

We are indebted to H. EPSTEIN, V. GLASER, J. H. LOWENSTEIN, B. SCHROER, K. SYMANZIK and W. ZIMMERMANN for communicating to us their work prior to publication, and for illuminating correspondence, as well as R. SEILER, for fruitful discussions.

([10]) J. H. LOWESTEIN: *Comm. Math. Phys.*, **24**, No. 1, 1 (1971), especially eqs. (3.4*d*), (3.4*e*), (3.5).

C H A P I T R E III

NOETHER'S THEOREM AND WARD IDENTITIES IN RENORMALIZABLE THEORIES

LETTERE AL NUOVO CIMENTO VOL. 4, N. 4 27 Maggio 1972

Noether's Theorem and Ward Identities in Renormalizable Theories.

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(ricevuto il 18 Febbraio 1972)

In a previous letter (1) referred to as I, we have exhibited Ward identities associated with space-time symmetries, in the general case of renormalizable field theories, understood in the perturbative sense. It is worth-while pointing out the main features of these identities, in their relationship with the application of Noether's theorem to the effective Lagrangian, in the sense of ZIMMERMANN (2) and LOWENSTEIN (3). These features are in fact quite independent from the symmetry group under consideration.

Given an effective Lagrangian $\mathscr{L}_{eff}(\varphi)$, and a symmetry group acting linearly on the fields φ, with the infinitesimal version

$$\varphi \to \varphi + t\,\delta\omega\varphi\,, \tag{1}$$

(*e.g.*, $t = \frac{1}{2}\partial$, in the case of space-time translations), one may consider Noether's current

$$J_\mu^N = \pi_\mu t\varphi\,, \tag{2}$$

where $\pi_\mu = \delta\mathscr{L}_{eff}/\delta\partial_\mu\varphi$, and where J_μ^N is assigned dimension three in the case of an internal symmetry group, four in the case of space-time translations.

The following Ward identity then holds:

$$\partial^\mu(\Omega, TJ_\mu^N(\xi)\varphi_1(x_1)\dots\varphi_n(x_n)\Omega) = \left(\Omega, T\frac{\delta\mathscr{L}_{eff}}{\delta\omega}(\xi)\varphi_1(x_1)\dots\varphi_n(x_n)\Omega\right) + \sum_{i=1}^{n}\delta(\xi - x_i)t_i(\Omega, T\varphi_1(x_1)\dots\varphi_n(x_n)\Omega)\,. \tag{3}$$

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(1) A. ROUET and R. STORA: *Lett. Nuovo Cimento*, **4**, 136 (1972).

(2) W. ZIMMERMANN: *Lectures on elementary partices and quantum field theory*, in *Brandeis University, Summer Institute in Theoretical Physics*, Vol. 1 (Cambridge, 1970).

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For instance, in the case of space-time translations,

$$\frac{\delta \mathscr{L}_{ett}}{\delta \omega} = \partial \mathscr{L}_{ett}(\xi) ,$$

so that $t_\mu^\nu = t_\mu^{(N)\nu} - g_{\mu\nu}\mathscr{L}_{ett}$ fulfills normal Ward identities (eq. (1) in I).

However, it is derivable to have Ward identities of the form

$$(4) \qquad \partial^\mu(\Omega, T\boldsymbol{J}_\mu(\xi)\varphi_1(x_1)\dots\varphi_n(x_n)\Omega) = \\ = (\Omega, T\boldsymbol{D}(\xi)\varphi_1(x_1)\dots\varphi_n(x_n)\Omega) + \sum \delta(\xi - x_i)\boldsymbol{t}_i(\Omega, T\varphi_1(x_1)\dots\varphi_n(x_n)\Omega) ,$$

where $\boldsymbol{D}(\xi)$ is an operator of given prescribed dimension and covariance. Equation (3) is in general not automatically in this form, except when the symmetry under consideration is exact. In the precess of expressing $\delta\mathscr{L}_{ett}/\delta\omega$, in eq. (3), in terms of a basis of operators of given dimension, there appear equal time terms [3] which spoil the canonical coefficients $\delta(\xi - x_i)\boldsymbol{t}_i$. Thus, in the case of broken symmetries, Noether's current cannot fulfill a canonical Ward identity such as (4). In the case of space-time symmetries, once Ward identities are fulfilled for the stress-energy tensor, the trace identity, connected with « broken » dilatation invariance, involves in general non canonical dimensions. This is connected with the well-known fact that the breaking is in this case of dimension four. However, in the case of internal symmetries, with breaking strictly smaller than four, it is indeed possible to construct a current with dimension not exceeding three, so that a canonical Ward identity (4) is fulfilled. This current will however, in general, not be Noether's current.

The details of the corresponding calculations will be given elsewhere, as well an extension to the case of correlation functions involving several currents and the corresponding generalization of Zimmermann's formalism.

* * *

The point of view we have adopted here concerning Ward identities owes much to an exchange of ideas with K. SYMMANZIK. We also wish to thank J. H. LOWENSTEIN and W. ZIMMERMANN for illuminating correspondence and for communicating their work prior to publication.

C H A P I T R E IV

RENORMALIZABLE MODELS WITH BROKEN SYMMETRIES

RENORMALIZABLE MODELS WITH BROKEN SYMMETRIES

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RENORMALIZABLE MODELS WITH BROKEN SYMMETRIES

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I. GENERAL REVIEW OF PERTURBATIVE RENORMALIZATION

There exist at present several versions[1] of renormalized pertur - bation theory, which develop in various directions some of the ideas contained in the work of N.N. Bogoliubov[2] and coworkers. The general set up, as recently described by H. Epstein and V. Glaser[1, 3] is as follows:

$\mathcal{F}$: Fock space of a family of free fields $\Phi_0(x)$

$U(a,A)$: the corresponding Fock representation of the covering of the Poincaré group

Ω : the vacuum state

$\underline{\mathcal{L}}_0(x)$: the "interaction" Lagrangian, a family of Wick monomials in $\underline{\Phi}$ and its derivatives which, together with a given monomial, contains all its submonomials.

$\underline{g}(x)$ a corresponding family of smooth space time dependent coupling constants, with fast decrease at infinity.

One can construct by recursion an operator valued formal power series in g:

$$S(\underline{g}) = \mathbb{1} + \sum_{n=1}^{\infty} \frac{(i)^n}{n!} \int dx_1 \dots dx_n \, T(\underline{x}_1, \dots \underline{x}_n)\, \underline{g}(x_1) \dots \underline{g}(x_n)$$

whose coefficients $T(\underline{x}_1, \dots \underline{x}_n) \overset{\text{Def.}}{=} T(\underline{\mathcal{L}}_0(x_1) \dots \underline{\mathcal{L}}_0(x_n))$ are properly defined "time ordered" products of the Poincaré covariant interaction Lagrangians, densely defined in $\mathcal{F}$ as operator valued distributions. The antitime ordered products are defined as coefficients of the formal power series for $S^{-1}(g)$:

$$S^{-1}(\underline{g}) = \mathbb{1} + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int dx_1 \dots dx_n \, \overline{T}(\underline{x}_1, \dots, \underline{x}_n)\, \underline{g}(x_1) \dots \underline{g}(x_n)$$

The recursive construction can be so carried out that the following causal factorization property holds:

$$S(\underline{g}_1 + \underline{g}_2) = S(\underline{g}_1)\, S(\underline{g}_2)$$

$$\text{if } \operatorname{supp} \underline{g}_1 \gtrsim \operatorname{supp} \underline{g}_2$$

$$\text{i.e. } (\operatorname{supp} \underline{g}_1 + \overline{V}_+) \cap \operatorname{supp} \underline{g}_2 = \emptyset$$

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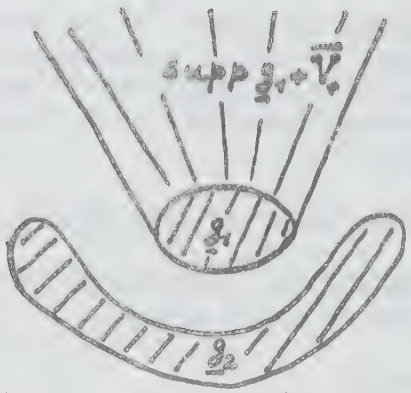

Fig. I Supp. $g_1 \supseteq$ Supp. g_2

At each step of the recursive construction, there is an ambiguity whose minimality can be discussed via a precise theory of power counting. This ambiguity arises solely from the fact that causality and spectrum force the fields, and therefore their correlation functions, to be distributions. The problem to be solved concerns a certain "discontinuity" C_n which, in view of the causality assumption, has the union of two opposite closed cones $\Gamma_n^{\pm}$ as x-space support. One has to express C_n as the difference of a retarded and advanced distribution with respective supports Γ_n^+, Γ_n^-. This can be done, but not by mere multiplication by some Heaviside step function, and the solution is in general non-unique. At the nth step there arises an ambiguity in the definition of the operator valued distribution $T(x_1, \dots, x_n)$ which is of the type

$$\Delta T(x_1, \dots x_n) = \sum_i P_i(\partial)\, \delta(x_1 - x_2) \dots \delta(x_{n-1} - x_n) : Q_{oi} :$$

where Q_{oi} is a Wick monomial in φ_o and its derivatives, $P_i(\partial)$ a monomial of derivatives, and the sum ranges over all possible terms compatible with Wick's theorem such that

$$\deg. P_i + \omega(Q_{oi}) \geq \omega_n = \sum_{i=1}^{n} (\omega(\ell_i) - 4) + 4 .$$

The index ω assigned to a Wick monomial adds up the number of derivatives and the number of fields each of which weighted by its naive dimension ($\frac{2j+2}{2}$ for a field carrying spin j if no better estimate holds due to the presence of an indefinite metric in $\mathcal{F}$).

Given two solutions, S and $\hat{S}$, characterized by coefficients $T(x_1 \dots x_n)$, $\hat{T}(x_1 \dots x_n) = T(x_1 \dots x_n) + \Delta T(x_1 \dots x_n)$, $\hat{S}$ is obtained by applying the prescription "T" to the interaction term

$$\int \hat{\mathcal{L}}_0(x,\underline{g})\,dx = \int \underline{\mathcal{L}}_0(x)\,\underline{g}(x)\,dx + \sum_{n=2}^{\infty} \frac{(i)^n}{n!} \int dx_1 \dots dx_n \quad \dots$$

$$\dots \Delta T(x_1, \dots x_n)\ \underline{g}(x_1) \dots \underline{g}(x_n)$$

Each operator time ordered product is expressed in terms of c-number distribution coefficients by means of Wick's theorem:

$$T\left(\mathcal{L}_{0,a_1}(x_1) \dots \mathcal{L}_{0,a_n}(x_n)\right) = \sum \left(\Omega, T\left(\mathcal{L}_{0,a_1}^{r_1}(x_1), \dots \mathcal{L}_{0,a_n}^{r_n}(x_n)\right)\Omega\right) \dots$$

$$\dots : Q_{0,a_1}^{r_1}(x_1) \dots Q_{0,a_n}^{r_n} : C_{r_1, \dots r_n} \ .$$

where $: \mathcal{L}_{0,a_i}^{r_i}\ Q_{0,a_j}^{r_j} : = \mathcal{L}_{0,a_i}$ and $C_{r_1 \dots r_n}$ are some combinatorial factors. Thus, in p-space, the vacuum expectation values of time ordered products are ambiguous up to polynomials of degrees determined by power counting. If all components of $\underline{\mathcal{L}}_0$ carry a non-vanishing mass, a "central" solution can be defined, which vanishes at zero momentum together with its ω first derivatives, ω being the power counting index. Non minimal solutions can be defined by assigning to some components of $\underline{\mathcal{L}}_0$ power counting indices larger than the naive index previously defined.

Physics usually requires to define an adiabatic limit which we now describe. Let us separate the components of $\underline{g}$ into two groups

$$\underline{\mathcal{L}}_0 \cdot \underline{g} = \underline{\mathcal{O}}_0 \cdot \underline{J} + \underline{\mathcal{L}}_0^{int} \cdot \underline{\lambda}$$

and let us inquire in what sense we can let the coupling constants $\underline{\lambda}$, whereas the space time varying source functions $\underline{J}$ will be used to generate fields and composite operators, such as currents, etc. The first term of $\underline{\mathcal{O}}$ as power series in $\underline{\lambda}$ is given by $\underline{\mathcal{O}}_0$. One can see that as $\underline{\lambda}(x) \to \underline{\lambda}$ the expression

$$\left(\Omega, S(J,\underline{\lambda})\Omega\right) / \left(\Omega, S(0,\underline{\lambda})\Omega\right)$$

has a limit whose coefficients of powers of $\underline{J}$ are the Green's functions of the interacting operators. It is, however, only when suitable normalization conditions are satisfied - whereby the vacuum to vacuum transition is so normalized that $(\Omega, S(0,\underline{\lambda})\Omega) \to 0$ as $\underline{\lambda}(x) \to \underline{\lambda}$, and the two point function is so defined that the one particle pole is fixed at its Fock space value[3] - that $S(J,\underline{\lambda})$ has a limit in the operator sense

as $\lambda(x) \to \lambda$. In many cases one does not want to do this, but rather constructs a theory of Green's functions which, after the adiabatic limit is taken, leads to an interpretation in a Fock space $\mathcal{F}_{phys.}$ different from the one given initially. This point of view is often to be taken in the description of systems with a broken symmetry as we shall see later.

The connection with conventional methods[2] which use regularization procedures is the following: Let ϕ_0^{reg} be a regularized free field defined in a larger Fock space $\mathcal{F}^{reg.} \supset \mathcal{F}$ and $\mathcal{L}_0^{reg.}(\phi_0^{reg})$ a regularized form of the interaction Lagrangian. Under certain conditions, (Pauli Villars regularization : $\mathcal{L}_0^{reg}(\phi_0^{reg}) = \mathcal{L}_0(\phi_0^{reg})$), it is possible to show that the central solution of the regularized theory goes over to the central solution of the non regularised theory in the sense of distributions, as the regularization is removed. It would be worthwhile to complete such a proof in more general cases, including that of the n-dimension regularization[4].

A particular solution of the regularized theory can easily be constructed in terms of Feynman graphs by mere multiplication of regularized Feynman propagators, and the corresponding central solution is uniquely determined by direct calculation of the regulator dependent Taylor expansions around p = 0 which can be gathered together into a redefinition of the interaction Lagrangian modulo regulator dependent ("infinite") counter terms.

Of particular interest are renormalizable theories for which $\omega(\mathcal{L}_0^{int}) \leq 4$ in which case possible counter terms occur in finite number, with coefficients formal power series in the coupling constants λ . In all the following, we shall limit ourselves to the study of renormalizable models. In particular, the next chapter will be devoted to a detailed description and comparison of the various, possibly non central as well as non minimal, solutions, in the adiabatic limit. The combinatorial identities which will be established will be the main tools in the study of the models treated in the later chapters.

CHAPTER I: REFERENCES AND FOOTNOTES

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II. DYNAMICS OF RENORMALIZABLE MODELS IN PERTURBATION THEORY

Within the framework of perturbation theory we will discuss the dynamical properties of renormalizable models based on the classical Lagrangian

$$\mathcal{L}_{cl.} = \mathcal{L}_{cl.0} + \mathcal{L}_{cl.}^{int} \qquad (II.1)$$

$\mathcal{L}_{cl.0}$ denotes the free Lagrangian. The interaction part $\mathcal{L}_{cl.}^{int}$ is written as

$$\mathcal{L}_{cl.}^{int} = \sum_{j=1}^{J} \lambda_j \mathcal{L}_{cl.j} \qquad (II.2)$$

where the coupling terms $\mathcal{L}_{cl.j}$ are monomials in the field components and their first derivatives. The Lagrangian is renormalizable if the naive dimension ω_j of each coupling term $\mathcal{L}_{cl.j}$ satisfies

$$\omega_j \leq 4$$

The components of all fields will be collected by a single field vector

$$\underline{\phi} = (\phi_1, \dots, \phi_A)$$

The construction of field operators and Green's functions by the method of Epstein and Glaser (I.1) was described in the previous Chapter. The advantage of this method is that with mathematical rigour the field operators are constructed recursively in perturbation theory such as to satisfy the fundamental principles of quantum field theory. For practical reasons, however, other methods of renormalization are more convenient which work in the adiabatic limit right from the beginning. In the work that follows we will use such an alternative method which is based on a renormalized version of the Gell-Mann Low formula[1]. First, we assign a degree δ_j satisfying

$$\omega_j \leq \delta_j \leq 4$$

to each coupling term $\mathcal{L}_{cl.j}$. This assignment will determine the number of subtractions to be used for separating the finite part of a Feynman integral. The time ordered Green's functions are constructed by

$$\langle T\, \phi_{\alpha_1}(x_1), \dots \phi_{\alpha_m}(x_m) \rangle = \langle T\, e^{i \int \mathcal{L}_{eff.0}^{int}(z)\, dz}\, \phi_{\alpha_1,0}(x_1) \dots \phi_{\alpha_m,0}(x_m) \rangle^{norm.} \qquad (II.3)$$

The superscript "horm" indicates that vacuum diagrams (disconnected closed loops) should be omitted. $\mathcal{L}^{int}_{eff,0}$ denotes the effective interaction part of the Lagrangian and is given by

$$\mathcal{L}^{int}_{eff,0} = \sum \lambda_j N_{j_j} [\mathcal{L}_{el.j_0}] \tag{II.4}$$

The subscript (0) indicates that the free field propagators pertaining to $\mathcal{L}_{el.0}$ are used to perform contractions. Time ordered functions involving the symbol N are defined as follows. The expansion of (II.3) with respect to powers of λ_j leads to expressions of the form

$$\langle T \phi_{\alpha_1,0}(x_1) \dots \phi_{\alpha_m,0}(x_m) N_{\rho_1}[M_{1,0}(u_1)] \dots N_{\rho_n}[M_{n,0}(u_n)] \rangle \tag{II.5}$$

where the M_k may be any of the coupling terms $\mathcal{L}_{\omega j}$. We may as well consider the more general case of arbitrary non-linear monomials $M_{j,0}$ in the free field components and their derivatives. The integers ρ_j are restricted by

$$\rho_j \geq d_j$$

where d_j is the naive dimension of the monomial $M_{j,0}$. ρ_j is called the degree assigned to the monomial $M_{j,0}$. (II.5) represents a time ordered Green's function of the Wick products

$$:M_{k,0}:(u_k)$$

While the Wightman functions of Wick products (II.5) are unique, the time ordered functions are only defined up to contact terms. The symbols $N_{\rho_1} \dots N_{\rho_n}$ serve to specify a unique choice of time ordered functions, yet to be defined. As has been emphasized by Bogoliubov the renormalization of the Gell-Mann Low formula rests upon a proper definition of the time ordered functions of free fields and their Wick products.

Without going into details we give a rough sketch of the definition which is rather involved for arbitrary diagrams due to the phenomenon of overlapping divergencies. One first expands the formal expression

$$\langle T \phi_{\alpha_1,0}(x_1) \dots \phi_{\alpha_m,0}(x_m) M_{1,0}(u_1) \dots M_{n,0}(u_n) \rangle \tag{II.6}$$

with respect to Feynman diagrams using Wick's rules. Each diagram has m external lines labeled by $x_1, \ldots, x_m$ and n vertices at $u_1, \ldots, u_n$. To each diagram Γ belongs a Feynman integral which in general diverges. We amputate the external lines and denote the unrenormalized Feynman integral in momentum space by

$$\text{(II.7)} \qquad \lim_{\varepsilon \to +0} \int dk_1 \ldots dk_t \; I_\Gamma (k_1 \ldots k_t ; p_1, \ldots p_m ; q_1, \ldots q_n)$$

The external momenta p_a, q_b correspond to the coordinates x_a, u_b. The momenta $k_1, \ldots, k_t$ form a set of independent integration variables. The unrenormalized integrand I_Γ is given by the usual Feynman rules pertaining to the Lagrangian (II. 1). The time ordered function (II.6) is defined by the finite part of (II.7) summed over all possible diagrams. The finite part will be formed by taking subtractions of the integrand I_Γ. To this end we introduce the degree

$$\text{(II.8)} \qquad \delta(\Gamma) = 4 - B - \frac{3}{2} F + \sum_{j=1}^{n} (\delta_j - 4)$$

of proper diagrams Γ. B is the number of external boson lines, F the number of external fermion lines. We recall that δ_j is the degree assigned to the vertex at u_j. If the degrees δ_j are chosen to be the naive dimensions d_j

$$\text{(II.9)} \qquad \delta_j = d_j$$

$\delta(\Gamma)$ is just the superficial divergence $d(\Gamma)$ of the diagram

$$\text{(II.10)} \qquad d(\Gamma) = 4 - B - \frac{3}{2} F + \sum_{j=1}^{n} (d_j - 4)$$

One has always

$$\delta(\Gamma) \geq d(\Gamma)$$

A proper diagram is called primitive if all proper subdiagrams have negative degree. For a primitive diagram the finite part is given simply by

$$\text{(II.11)} \qquad \lim_{\varepsilon \to +0} \int dk_1 \ldots dk_t \; (1 - t^{\delta(\Gamma)}_{p,q}) \cdot I_\Gamma (k_1, \ldots k_t ; p_1, \ldots p_m ; q_1, \ldots q_n)$$

Here $t^{\delta(\Gamma)}_{p,q}$ denotes the Taylor operator in p, q up to order $\delta(\Gamma)$. If $\delta(\Gamma) < 0$ we set $t^{\delta(\Gamma)}_{p,q} = 0$. If the degrees equal the dimensions (see (II.9-10)) we precisely have Dyson's prescription for separating the finite part. In the general case

oversubtractions are made which would not be required to render the integral finite. For arbitrary proper diagrams overlapping divergencies may occur which can be removed by applying Bogoliubov's combinatorial method (I.2). The finite part then is of the form

$$(II.12)\quad \lim_{\varepsilon \to +0} \int dk_1 \dots dk_L \left(1 - \xi_{p,q}^{\delta(\gamma)}\right)(\dots) \cdot \cdot I_\Gamma\,(k_1, \dots k_L; p_1, \dots p_n; q_1, \dots q_m) \cdot$$

where (...) indicated subtractions for subdiagrams. For details we refer to ref. [1] . This method of separating the finite part from a Feynman integral is thus an extension of the original work of Dyson and Salam. It can further be shown that the finite parts (II.12) equal those derived by Bogoliubov and Hepp with a somewhat different method[3] . Their results in turn are contained in the general class of solutions which was obtained by Epstein and Glaser in the adiabatic limit [I.3] .

As physical parameters of the theory we distinguish mass constants and coupling constants. Mass constants occur as coefficients of the free Lagrangian and are related to the discrete spectrum of the mass operator $P_\mu P^\mu$. The coupling constants may suitably be defined through various Green's functions. Let $m_1, \dots, m_A$ and $g_1, \dots, g_B$ be independent sets of mass or coupling constants which completely characterize the physics of the model. A corresponding set of renormalization conditions is then imposed on the Green's functions.

The renormalization conditions recursively determine $\lambda_1, \dots, \lambda_J$ as power series in $g_1, \dots g_B$ with finite, mass dependent coefficients. If all parameters λ_j vanish in zero order the expansion of (II.3) with respect to Feynman diagrams provides the power series expansion in g_j. However, if some λ_j do not vanish in zero order a series has to be summed for obtaining a finite order of perturbation theory. This difficulty is characteristic for some models with broken symmetries. A generalization of the present treatment will be proposed below in this Chapter which allows to avoid the summation problem.

We next turn to the definition of composite operators[1]. Let M be a non-linear monomial in the field components and their derivatives. We will construct a sequence

$$(II.13)\quad N_a[M(y)]\,, \quad a = d, d+1, d+2, \dots$$

of composite operators which are associated with this monomial. d is the naive dimension of M. Green's functions involving composite operators are defined by the following extension of (II.3)

$$T$$

$$(II.14)\quad \langle T\, \mathfrak{Z} \rangle = \langle T\, e^{\,i\int \mathcal{L}^{int}_{eff,0}(z)\,dz}\, Z_0 \rangle^{norm.}$$

with the abbreviation

$$(II.15)\quad \mathfrak{Z} = \phi_{a_1}(x_1)\cdots\phi_{a_m}(x_m)\, N_{a_1}[M_1(y_1)]\cdots N_{a_n}[M_n(y_n)]$$

We now formulate a generalization of the formalism which is useful for models with broken symmetries [14]. In particular, it provides a simple remedy for the summation problem of such models. This will be demonstrated in Chapter IV on the Goldstone and the Higgs models. In this generalization we let the classical Lagrangian be a polynomial in a new parameter s

$$(II.16)\quad \mathcal{L}_{cl}(s) = \mathcal{J}_4 + s\mathcal{J}_3 + s^2\mathcal{J}_2 + s^3\mathcal{J}_1$$

The naive dimensions d of the monomials occurring in $\mathcal{J}_\alpha$ are required to be less than or equal to

$$d \leq \alpha$$

Accordingly $\mathcal{J}_4$ is linear in the fields. The parameter s is allowed to vary within

$$(II.7)\qquad 0 \leq s \leq 1$$

While s = 1 is the physically relevant case the dependence on s will play an important part in the subtraction procedure.

Before writing down the analogue of (II.14) we have to clarify the definition of the free Lagrangian which becomes less trivial in this generalization. The physical mass and coupling constants of the theory refer to the case s = 1. Accordingly, renormalization conditions are also imposed at s = 1. As usual, the free Lagrangian $\mathcal{L}_{cl,0}$ at s = 1 is taken to be the Lagrangian of the incoming and outgoing fields, apart from possible normalization factors. For general s we define the free Lagrangian by the sum of all bilinear terms of $\mathcal{L}_{cl}(s)$ with the coefficients replaced by their zero order values. The dependence of these coefficients on the coupling constants will be discussed shortly. For the moment we only note that the coefficients of $\mathcal{L}_{cl,0}(s)$ are quadratic functions of s, due to (II.16).

The Green's functions of the theory are constructed by

$$(II.18)\quad \langle T Z \rangle = \langle T_s \, e^{i \int N_4 [\mathcal{L}^{int}_{d,0}(s,x)]\, dx} Z_0 \rangle^{norm.}$$

with

$$(II.19)\quad \mathcal{L}^{int}_{d}(s) = \mathcal{L}_{d}(s) - \mathcal{L}_{d,0}(s) = \sum \lambda_j \mathcal{L}_{d,j}(s)$$

and

$$(II.20)\quad Z = \phi_{d_1}(x_1) \cdots \phi_{d_m}(x_m)\, N_{a_1}[\mathcal{O}_1(y_1)] \cdots N_{a_n}[\mathcal{O}_n(y_n)]$$

The formula defines the time ordered Green's functions of the fields, as well as Green's functions of composite operators. The symbol T_s indicates a special time ordering which we are going to define now. The expansion of (II.18) leads to expressions of the form

$$(II.21)\quad \langle T_s\, \phi_{d_1}(x_1) \cdots \phi_{d_m}(x_m)\, N_{p_1}[M_1(u_1)] \cdots N_{p_r}[M_r(u_r)] \rangle$$

In defining (II.21) we proceed as before but replace (II.12) by

$$(II.22)\quad \lim_{\varepsilon \to +0} \int dk_1 \cdots dk_t \,(1 - t^{\delta(\Gamma)}_{p,q,s})(\cdots) \cdot I_\Gamma(s; k_1, \cdots k_t; p_1, \cdots p_m; q_1, \cdots q_r)$$

The new feature is that now subtractions are made in p_k, q_l and s simultaneously. In other words, s is treated like an external momentum, as far as subtractions indicated by (...) are modified similarly[5]. Finally, the renormalization conditions imposed at $s = 1$ determine the parameters λ_j as power series in g_i with mass dependent coefficients. Let $m_1, \ldots, m_A, g_1, \ldots, g_B$ be complete sets of independent mass and coupling constants at $s = 1$ [6]. Through the renormalization conditions the parameters $\lambda_1, \ldots \lambda_J$ become power series in the g_i with finite coefficients which depend on the mass constants[7]. Without proof we will now state some dynamical properties of Green's functions and field operators which hold to any order in perturbation theory. The dynamics is governed by the effective Lagrangian

$$(II.23)\quad \mathcal{L}_{eff}(s) = N_4(J_4) + s\, N_3[J_3] + s^2 N_2(J_2) + s^3 J_1 \,.$$

The principal dynamical laws can be stated as follows.

I. Action principle [2, 8]

$$\frac{\partial}{\partial \xi} \langle T \prod_j \phi_{\nu_j}(x_j) \rangle = i \langle T \int \frac{\partial \mathcal{L}_{eff}(z)}{\partial \xi} dz \prod_j \phi_{\nu_j}(x_j) \rangle \tag{II.24}$$

ξ is an arbitrary parameter. In $\partial \mathcal{L}_{eff} / \partial \xi$ the differential operator acts only on the coefficients of the field monomials.

II. Space-time differentiation of normal products [2].

$$\partial_\mu N_\delta[M] = N_{\delta+1}[\partial_\mu M] \tag{II.25}$$

$$\partial_\mu^z \langle T N_\delta[M(z)] \prod_j \phi_{\nu_j}(x_j) \rangle = \langle T N_{\delta+1}[\partial_\mu^z M(z)] \prod_j \phi_{\nu_j}(x_j) \rangle \tag{II.25}$$

III. Linear relations among normal products [1, 2].

If $\varphi < \delta$, a given normal product of degree φ and its Green's functions may be written as linear combinations

$$N_\varphi[M] = \sum a_i N_\delta[M_i] \tag{II.27}$$

$$\langle T N_\varphi[M(z)] \prod_j \phi_{\nu_j}(x_j) \rangle = \sum_i a_i \langle T N_\delta[M_i(z)] \prod_j \phi_{\nu_j}(x_j) \rangle \tag{II.28}$$

where the sum extends over all monomials M_i of dimensions $d \leq \delta$ with appropriate quantum numbers and invariance properties.

IV. Equations of motion

$$\frac{\delta \mathcal{L}_{eff}}{\delta \phi_k} - \partial^\mu \frac{\delta \mathcal{L}_{eff}}{\delta \partial^\mu \phi_k} = 0 \tag{II.29}$$

$$\langle T\left[\frac{\delta \mathcal{L}_{eff}}{\delta \phi_k}(z) - \partial^\mu \frac{\delta \mathcal{L}_{eff}}{\delta \partial^\mu \phi_k}(z)\right] \prod_j \phi_{v_j}(x_j)\rangle =$$

(II.30)

$$i \sum_j \delta_{k,j}\, \delta(z - x_j) \langle T \prod_{j \neq k} \phi_{v_j}(x_j)\rangle$$

In $\delta \mathcal{L}_{eff}/\delta \phi_k$, $\delta \mathcal{L}_{eff}/\delta \partial^\mu \phi_k$ the action of the differential operators on normal products is defined by

(II.31)
$$\frac{\delta}{\delta \phi_k} N_a[M] = N_{a-d_k}\left[\frac{\delta M}{\delta \phi_k}\right]$$

$$\frac{\delta}{\delta \partial^\mu \phi_k} N_a[M] = N_{a-d_k-1}\left[\frac{\delta M}{\delta \partial^\mu \phi_k}\right]$$

V. Equations of motion multiplied by a field component[2]

(II.32)
$$\phi_\ell \left[\frac{\delta \mathcal{L}_{eff}}{\delta \phi_k} - \partial^\mu \frac{\delta \mathcal{L}_{eff}}{\delta \partial^\mu \phi_k}\right] = 0$$

(II.33)
$$\langle T\, \phi_\ell \left[\frac{\delta \mathcal{L}_{eff}}{\delta \phi_k} - \partial^\mu \frac{\delta \mathcal{L}_{eff}}{\delta \partial^\mu \phi_k}\right](z), \prod_j \phi_{v_j}(x_j)\rangle =$$

$$i \sum_j \delta_{kj}\, \delta(z - x_j) \langle T\, \phi_\ell(x_j) \prod_{j \neq k} \phi_{v_j}(x_j)\rangle$$

Here multiplication of a normal product by a field component at the same point is defined by

(II.34)
$$\phi_\ell(z)\, N_a[M(z)] = N_{a+d_\ell}[\phi_\ell(z)\, M(z)]$$

VI. Equations of motion multiplied by a field monomial[9]

Let Q(x) be a monomial of dimension d in the fields and their derivatives. Then

(II.35)
$$\{Q\} \cdot \left[\frac{\delta \mathcal{L}_{eff}}{\delta \phi_k} - \partial^\mu \frac{\delta \mathcal{L}_{eff}}{\delta \partial^\mu \phi_k}\right] = 0$$

(II.36)
$$\langle T\{Q\}\cdot\left[\frac{\delta\mathcal{L}_{eff}}{\delta\phi_k}-\partial^\nu\frac{\delta\mathcal{L}_{eff}}{\delta\partial^\nu\phi_k}\right](x)\prod_j\phi_{\nu_j}(x_j)\rangle=$$
$$=\sum_j\delta_{kj}\,\delta(x-x_j)\langle T\,N_{\lambda}[Q(x_j)\prod_{\ell\neq j}\phi_{\nu_\ell}(x_\ell)\rangle$$

$\{Q\}$ acting on a normal product of a monomial M denotes the anisotropic normal prodct

(II.37) $$\{Q\}\cdot N_a[M]=N_{a+d}[\{Q\}M]$$

which is defined in the following way. Let dim M denote the dimension of M. The Green's functions of (II.37) are again constructed by (II.22) with the subtraction degree $\delta(\gamma)$ assigned to each proper subgraph γ given by (II.10). However, the degree δ assigned to the normal product vertex is defined in an anisotropic manner. δ is equal to its minimal value, d + d(M), if γ has no internal lines arising from contractions with fields in M, and is equal to d + a otherwise. It should be noted that $N_{a+d}[\{Q\}M]$ can always be expanded in terms of isotropic normal products of degree a + d using relations similar to those of III.

Finally we discuss the connection between formulations involving the parameter s and the conventional treatment given in the beginning of this chapter. We start with the simplest case that the free Lagrangian of the s-formulation is independent of s. Then the rules for constructing Feynman integrals involve s only in form of powers assigned to some vertices of the diagram. s never occurs in the masses of the Feynman denominators. Analyzing the subtraction rules one finds that the model based on the s-dependent Lagrangian

(II.38)
$$\mathcal{L}_{cl}(s)=\mathcal{L}_{cl,0}(s)+\mathcal{L}_{cl}^{int}(s)$$
$$\mathcal{L}_{cl,0}(s)\ \text{independent of } s$$
$$\mathcal{L}_{cl}^{int}(s)=\mathcal{G}_4+s\,\mathcal{G}_3+s^2\mathcal{G}_2+s^3\mathcal{G}_1$$

is equivalent to the model based on the conventional interaction Lagrangian

(II.39) $$\mathcal{L}_{eff}^{int}=N_4[\mathcal{G}_4]+N_3[\mathcal{G}_3]+N_2[\mathcal{G}_2]+\mathcal{G}_1$$

Stated more precisely, the Green's functions evaluated from (II.18) with N_4 applied to the interaction part (II.38) are at s = 1 identical to the Green's functions defined through (II.14) with interaction part (II.39). Thus the generalization is not expected to give any new information for such cases.

If the unperturbed mass parameters depend on s the situation is quite different. Let $\mathcal{L}_{cl.,0}(s)$ and $\mathcal{L}_{cl}^{int}(s)$ be of the form

$$\mathcal{L}_{cl.,0}(s) = \mathcal{F}_4 + s\mathcal{F}_3 + s^2\mathcal{F}_2 \tag{II.40}$$

$$\mathcal{L}_{cl.}^{int}(s) = \mathcal{G}_4 + s\mathcal{G}_3 + s^2\mathcal{G}_2 + s^3\mathcal{G}_1 \tag{II.41}$$

with the coefficients of the monomials in $\mathcal{F}_\alpha$ independent of s and the coupling constants g_i. For the coefficients of the $\mathcal{G}_4$ it is assumed that they vanish in zero order of g_i. Again there is a formal equivalence between the s-dependent Lagrangian

$$\begin{aligned}\mathcal{L}_{cl}(s) &= \mathcal{L}_{cl.,0}(s) + \mathcal{L}_{cl.}^{int}(s) \\ &= \mathcal{J}_4 + s\mathcal{J}_3 + s^2\mathcal{J}_2 + s^3\mathcal{J}_1\end{aligned} \tag{II.42}$$

and the conventional Lagrangian

$$\mathcal{L}_{eff} = N_4[\mathcal{J}_4] + N_3[\mathcal{J}_3] + N_2[\mathcal{J}_2] + \mathcal{J}_1 \tag{II.43}$$

But it turns out that the appropriate form of the free Lagrangian is

$$\mathcal{L}_{eff,0} = N_4[\mathcal{F}_4 + \mathcal{F}_3 + \mathcal{F}_2] \tag{II.44}$$

so that the interaction part becomes

$$\begin{aligned}\mathcal{L}_{eff.}^{int} = {} & N_4[\mathcal{G}_4] + N_3[\mathcal{G}_3] + N_2[\mathcal{G}_2] + \mathcal{G}_1 \\ & + N_3[\mathcal{F}_3] - N_4[\mathcal{F}_3] \\ & + N_2[\mathcal{F}_2] - N_4[\mathcal{F}_2]\end{aligned} \tag{II.45}$$

The coefficients of $\mathcal{F}_2$ and $\mathcal{F}_3$ are independent of the coupling constants g_j. As was discussed above this leads to a summation problem since an infinite number of Feynman integrals contributes to a finite order in g_j. Since $\mathcal{F}_3$ and $\mathcal{F}_4$ are bilinear in the fields one deals with geometric series in momentum space which diverge in parts of the integration domains. Thus the formalism based on the Lagrangian (II.44 - 45) is not satisfactory. With the original s-formulation, however, each order in the g_j is represented by a finite number of Feynman integrals.

Despite of this it is always possible to construct a Lagrangian of the conventional type which is equivalent to (II.42), but in a non-trivial way. The disadvantage of this Lagrangian is that symmetries which the s-dependent Lagrangian may display could be completely distorted for the equivalent conventional Lagrangian. This happens to be the case for the Goldstone and the Higgs model.

We briefly state and prove this equivalence theorem [10].

EQUIVALENCE THEOREM

Let

$$\mathcal{L}'_{cl}(s) = \mathcal{L}'_{cl,0}(s) + \mathcal{L}'^{\,int}_{cl}(s)$$

(II.46)

$$= \mathcal{J}'_4 + (s-1)\mathcal{J}'_3 + (s^2-1)\mathcal{J}'_2 + (s^3-1)\mathcal{J}'_1$$

$\mathcal{J}'_4$ are polynomials of their fields and their first derivatives with s-independent coefficients, the naive dimensions of the monomials in $\mathcal{J}'_4$ be less than or equal to four, the dimensions of monomials in $\mathcal{J}'_{1,2,3}$ be less than or equal to 1,2,3. Let mass and coupling constants be defined through renormalization conditions on the time ordered Green's functions of fields. Then there exists a Lagrangian

$$\mathcal{L}''_{cl} = \mathcal{L}''_{cl.0} + \mathcal{L}''^{\,int}_{cl.}$$

(II.47)

$$\mathcal{L}''^{\,int}_{eff} = N_4\left[\mathcal{L}''^{\,int}_{cl.}\right]$$

with the same renormalization conditions such that the models based on (II.46) and (II.47) are identical. More precisely, the time ordered functions of the fields are identical whether constructed by (II.18) with the Lagrangian (II.46) or by (II.3)

with (II.47). If the coefficients of the interaction part of (II.46) vanish in zero order of the coupling constants the same is true for (II.47). Hence no summation problem arises with (II.47).

Remark: Green's functions involving normal products may change. This does not impair the equivalence of the models but only indicates that similarly constructed normal products must not be identified.

Proof: In order to eliminate the s-dependent terms we will construct a family of Lagrangians

$$\mathcal{L}_{cl}(s) = \mathcal{J}_4 + (s-1)\mathcal{J}_3 + (s^2-1)\mathcal{J}_2 + (s^3-1)\mathcal{J}_1 \tag{II.48}$$

which includes the Lagrangian (II.46) with

$$\mathcal{J}_4 = \mathcal{J}_4' = \mathcal{L}_{cl}(1) \quad , \quad \mathcal{J}_\alpha = \mathcal{J}_\alpha' \qquad \alpha = 1,2,3 .$$

and a Lagrangian

$$\mathcal{L}_{cl}''(s) = \mathcal{J}_4'' \quad , \quad \mathcal{J}_1'' = \mathcal{J}_2'' = \mathcal{J}_3'' = 0$$

with no s-dependent terms. For this family it will be shown that the Green's functions of fields do not change.

Let

$$M_{d1}, \dots M_{d\,A(d)} \tag{II.49}$$

be all monomials of naive dimension d which can be formed out of the field components and their derivatives. We express $\mathcal{J}_4', \mathcal{J}_\alpha'$ as linear combinations of these monomials

$$\mathcal{J}_4' = \sum_{d=1}^{4} \sum_{j=1}^{A(d)} c_{dj}' M_{dj}$$

$$\mathcal{J}_\alpha' = \sum_{d=1}^{\alpha} \sum_{j=1}^{A(d)} t_{\alpha dj}' M_{dj} \qquad \alpha = 1,2,3$$

With this notation the original Lagrangian becomes

$$\mathcal{L}_{cl}'(s) = \sum_{d=1}^{4} \sum_{j=1}^{A(d)} c_{dj}' M_{dj} + \sum_{\alpha=1}^{3} (s^{4-\alpha}-1) \sum_{d=1}^{\alpha} \sum_{j=1}^{A(d)} t_{\alpha dj} M_{dj}$$

We now study the class

$$\mathcal{L}_{cl}(s) = \sum_{d=1}^{4} \sum_{j=1}^{A(d)} c_{dj} M_{dj} + \sum_{d=1}^{3} (s^{4-d}-1) \sum_{d=1}^{a} \sum_{j=1}^{A(d)} t_{adj} M_{dj} \tag{II.50}$$

of Lagrangians where the c_{dj} are functions of arbitrary parameters t_{adj}. Our intention is to place suitable restrictions on the functions c_{dj} such that at $s = 1$ the Green's functions remain independent of t_{adj}. To this end we form the derivative

$$\frac{\partial \langle T X \rangle}{\partial t_{\beta ek}} = i \sum_{d=1}^{a} \sum_{j=1}^{A(d)} \frac{\partial c_{dj}}{\partial t_{\beta ek}} \int dx \langle T N_4[M_{dj}(x) X] \rangle$$
$$+ i \int dx \langle T (N_\beta[M_{ek}(x)] - N_4[M_{ek}(x)]) X \rangle$$

(II.28) then implies that $\langle T N_\beta[M_{ek}] X \rangle$ is a linear combination of Green's functions involving N_4-products

$$\langle T (N_\beta[M_{ek}(x)] - N_4[M_{ek}(x)]) X \rangle =$$
$$= - \sum r_{\beta ek dj} \langle T N_4[M_{dj}(x)] X \rangle$$

Thus

$$\frac{\partial \langle T X \rangle}{\partial t_{\beta ek}} = 0$$

holds provided

$$\frac{\partial c_{dj}}{\partial t_{\beta ek}} = r_{\beta ek dj}$$

The $r_{\beta ek dj}$ depend on $t_{\beta dk}$ directly and through the functionals c_{dj}. A more detailed discussion shows that a solution exists with the initial values

$$c_{dj} = c'_{dj} \quad \text{at} \quad t_{\beta dj} = t'_{\beta dj}$$

With these solutions as coefficients c_{dj} the family (II.50) of Lagrangians has all desired properties. We finally set

$$t_{\beta dj} = 0$$

and find

$$\mathcal{L}''_{cl} = \sum_{j=1}^{A(d)} c''_{dj} M_{dj}$$

where c''_{dj} denotes the values at $t_{\beta dj} = 0$. Since the Green's functions of the basic fields do not change the same

normalization conditions are satisfied throughout. In zero order of the coupling constants the coefficients $r_{\beta e k d j}$ vanish. Hence the zero order values of the c_{dj} do not change. Accordingly, the free part of the Lagrangian (II.50) remains the same at $s = 1$ and the coefficients of the interaction part $\mathcal{L}''_{eff}$ vanish in zero order. This completes the proof.

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III. EXACT AND BROKEN SYMMETRIES: RENORMALIZATION EFFECTS.

In the following, we shall exclusively consider renormalizable effective Lagrangians whose dimension four kinematical parts involve non vanishing mass terms. Superrenormalizable couplings may otherwise be present and will be weighted by a power of the previously introduced parameter $\underline{s}$ which measures their degree of superrenormalizability. The presence of superrenormalizable interactions is, as we shall see, essential to the definition of broken symmetries [1]. In all cases, the symmetry or lack of symmetry is best described by the effective Noether theorem, which follows from the various effective equations of motion described in chapter II and is summarized by a set of Ward identities. We shall first, for the purpose of orientation, look at the simple case of space time symmetries, and then will go into the subject of internal symmetries.

1. Space time symmetries. The energy momentum tensor.

We recall the effective equations of motion:

$$\left\langle T\left(\frac{\delta \mathcal{L}_{eff}}{\delta \Phi} - \partial_\mu \frac{\delta \mathcal{L}_{eff}}{\delta \partial_\mu \Phi}\right)_{\mathcal{L}_{eff}}(x) X \right\rangle = i \sum_{k \in X} \delta(x - x_k) \langle T X_{/k} \rangle$$

where, for each term in $\mathcal{L}_{eff}$, the corresponding derivative with respect to φ has to be counted with a power counting degree diminished by dim φ. The Noether current associated with space time translations:

$$\varphi(x) \to \varphi(x+a) \simeq \varphi(x) + a^\mu \partial_\mu \varphi(x)$$

is

$$j_\mu^{Noeth.\,\nu} = \pi_\mu \partial^\nu \varphi$$

with

$$\pi_\mu = \frac{\delta \mathcal{L}_{eff}}{\delta \partial_\mu \varphi}$$

According to the preceding rules $j_\mu^{Noeth.\,\nu}$ has to be counted with dimension four. Using the effective equations of motion under the quadratic combination $\pi_\mu \partial_\nu \varphi$, we have:

$$\partial^\mu \langle T\, \pi_\mu \partial^\nu \varphi X \rangle = \langle T\, \partial^\mu \pi_\mu \partial^\nu \varphi X \rangle + \langle T \pi_\mu \partial^\mu \partial^\nu \varphi X \rangle$$

$$= \langle T \frac{\delta \mathcal{L}_{eff}}{\delta \varphi} \partial^\nu \varphi X \rangle + \langle T \frac{\delta \mathcal{L}_{eff}}{\delta \partial_\mu \varphi} \partial^\nu \partial_\mu X \rangle$$
$$-i \sum_{k \in X} \delta(x - x_k) \partial_{(k)}^\nu \langle TX \rangle$$
$$= \partial^\nu \langle T \mathcal{L}_{eff}(x) X \rangle + \sum_{k \in X} \delta(x - x_k) \frac{1}{i} \partial_{(k)}^\nu \langle TX \rangle$$

Hence, the canonical energy momentum tensor[2]

$$\Theta_\mu^{\nu,C} = \pi_\mu \partial^\nu \varphi - g_\mu^\nu \mathcal{L}_{eff}$$

fulfills the Ward identity

$$\partial^\mu \langle T \Theta_\mu^{\nu,C}(x) X \rangle = \sum_{k \in X} \delta(x - x_k) \frac{1}{i} \partial_{(k)}^\nu \langle TX \rangle$$

The Ward identity associated with Lorentz invariance usually requires the use of the Belinfante energy momentum tensor, of the form

$$\Theta_\mu^{\nu,B} = \Theta_\mu^{\nu,C} + \partial^\lambda f_{\mu\lambda}^\nu$$

where $f_{\mu\lambda}^\nu$ is given by the usual formula [3], is formally antisymmetric in μ, λ, so that it does not spoil the Ward identity and has to be assigned dimension 3 . The symmetry of $\Theta_{\mu\nu}^B$ in μ and ν being a consequence of the equations of motion, results into the asymmetry identity

$$\langle T \Theta_{\mu\nu}^B(x) X \rangle - \langle T \Theta_{\nu\mu}^B(x) X \rangle = \sum_{k \in X} \delta(x - x_k) S_{\mu\nu}^{(k)} \langle TX \rangle$$

where the $S_{\mu\nu}^{(k)}$ are the relevant spin matrices.

The asymmetry identity, and the Ward identity put together, result into the Ward identity for the Lorentz current:

$$M_{\lambda,\mu\nu} = x_\mu \Theta_{\lambda\nu}^B - x_\nu \Theta_{\lambda\mu}^B$$

$$\partial^\lambda \langle T M_{\lambda,\mu\nu}(x) X \rangle = \sum_{k \in X} \delta(x - x_k) \left[\frac{1}{i} \left(x_\mu^{(k)} \partial_\nu^{(k)} - x_\nu^{(k)} \partial_\mu^{(k)} \right) + S_{\mu\nu}^k \right] \cdot \langle TX \rangle$$

Finally, the dilation current which generates broken scale invariance can be conveniently written as

$$D_\lambda = x^\mu \Theta^{c.c.J.}_{\lambda\mu}$$

where the "improved" energy momentum tensor [4] of Callan, Coleman, Jackiw is given by

$$\Theta^{c.c.J.}_{\lambda\mu} = \Theta^{B}_{\lambda\mu} + \sum_s \frac{1+b_s}{6} \left(\Box g_{\mu\nu} - \partial_\mu \partial_\nu \right) \phi_s^2$$

where the sum ranges over scalar fields and the factors $(1+b_s)$ are the finite wave function renormalization factors occurring in $\mathcal{L}_{eff}$ in front of the quadratic terms $\frac{1}{2}\partial_\mu \phi_s \partial^\mu \phi_s$.

The corresponding Ward identity

$$\partial^\lambda \langle T\, D_\lambda(x)\, X \rangle = 2 \langle T\, M^2(x) X \rangle + \sum_{k \in X} \delta(x - x_{(k)}) \frac{1}{i} \cdot \cdot \left(x^k_{(k)} \partial^{(k)} + d^{(k)} \right) \langle T X \rangle$$

where $M^2(x)$ denotes the set of mass terms occurring in $\mathcal{L}_{eff}$, and thus contains dimension four terms, and $d^{(k)}$ is the canonical dimension of field $\varphi^{(k)}$, is a consequence of the Ward identity together with the trace identity:

$$\langle T\, \Theta^{\mu\, c.c.J.}_{\mu}(x)\, X \rangle = 2 \langle T\, M^2(x) X \rangle + \sum_{k \in X} \delta(x - x_k) \frac{1}{i} d^{(k)} \langle T X \rangle$$

The improvement thus does not yield a "soft" trace, and one can show that no other choice can be made so that the trace becomes "soft", at the expense of changing the canonical dimensions into abnormal dimensions [2,5]. By integrating the Ward identity for the dilation current or combining the trace identity with the Ward identity for the energy momentum tensor at zero momentum, one obtains the so called Callan Symanzik equation [2] after full use of the reduction formulae of chapter II.

2. Internal Symmetries.

Let $\mathcal{L}_{eff}$ be of the form

$$\mathcal{L}_{eff} = \mathcal{L}_{sym.} + \mathcal{L}_{break.}$$

where the terms in $\mathcal{L}_{break.}$ have dimensions strictly less than four and all mass terms of dimension four in $\mathcal{L}_{sym.}$ are non vanishing. Symmetry and symmetry breaking are understood with respect to a transformation of the type

$$\underline{\varphi} \to {}^{g}\underline{\varphi} = \mathcal{D}(g)\,\underline{\varphi} \quad , g \in G \ .$$

where G denotes a compact Lie group. $\mathcal{D}$ a finite dimensional unitary representation of G. The infinitesimal transformation law is of the type

$$\underline{\varphi} \to \underline{\varphi} + i\,\vec{\theta}.\delta\vec{\omega}\,\underline{\varphi}$$

where $\vec{\theta}$ represents the Lie algebra $\mathcal{G}$ of G in $\mathcal{D}$. When $\mathcal{D}$ is the adjoint representation, we shall denote by $\vec{t}$ the representatives of $\mathcal{G}$. Defining

$$\vec{J}^{\,\text{Noeth.}}_{\mu} = i N_3\, \pi_\mu\, \vec{\theta}\, \varphi$$

and using the effective equations of motion, we obtain the Ward identity

$$\partial^{\mu} \langle T\, \vec{J}^{\,\text{Noeth.}}_{\mu}(x)\, X \rangle = \langle T\, \frac{\delta \mathcal{L}_{\text{break}}}{\delta \varphi_i}(x) X \rangle + \sum_{k \in X} \delta(x - x_k)\, \vec{\theta}_k \langle TX \rangle$$

which has to be understood, as the whole theory in the sense of multiple formal series in all the parameters of $\mathcal{L}_{\text{break}}$ and some of the parameters of $\mathcal{L}_{\text{sym}}$. One could similarly establish current algebra Ward identities[6] which assume a normal form provided a suitable time ordered product involving two currents is defined, which differs from the time ordered products we have used so far by allowed ambiguities.

It is clear that the treatment given so far calls for resummation procedures which allow a decent interpretation within the physical Fock space alluded to in chapter I. This can be done as follows. First of all, in general $\mathcal{L}_{\text{break}}$ will induce nonvanishing field vacuum expectation values $\langle \underline{\varphi} \rangle = \underline{F}$. One can first perform the change of variable

$$\underline{\varphi}' = \underline{\varphi} - \underline{F} \qquad \langle \varphi' \rangle = 0$$

One can show[7] that the Green functions for $\underline{\varphi}'$ can be computed from the Lagrangian

$$\mathcal{L}'(\underline{\varphi}') = \mathcal{L}(\underline{\varphi}' + \underline{F})$$

with the dimension assignment given by

$$N_s(\varphi'+F)^p = \sum_q \binom{p}{q} F^q N_{s-1,q} \varphi'^{p-q}$$

and the Ward identity reads

$$\partial^\mu \langle T J'_\mu(x) X' \rangle = \langle T \vec{D}'(x) X' \rangle + \sum_{k \in X} \delta(x-x_k) \vec{\theta}_k \langle T X' \rangle + \sum_k \delta(x-x_k) (\vec{\theta} F)_k \langle T X'_{\hat{k}} \rangle$$

where J'_μ and $\vec{D}'$ are respectively deduced from J^{Noeth}_μ, $\frac{\delta \mathcal{L}^{break}}{\delta \omega}$ by the substitution rule $\varphi \to \varphi' + F$ according to the above formula. The introduction of the s parameter and its use as indicated in Chapter II allow to perform the necessary resummations. On the other hand, the equivalence theorem described in chapter II then asserts the existence of a dimension four effective Lagrangian (without a linear term) which yields the Green's functions $\langle T X' \rangle$. One should however be careful that composite operators J_μ, $\vec{D}$ such that

$$\langle T J_\mu(x), X' \rangle_{\mathcal{L}_4} = \langle T J'_\mu(x) X' \rangle_{(\mathcal{L}_{sym} + \mathcal{L}_{break})(\varphi'+F)}$$

$$\langle T \vec{D}(x) X' \rangle_{\mathcal{L}_4} = \langle T \vec{D}'(x) X' \rangle_{(\mathcal{L}_{sym} + \mathcal{L}_{break})(\varphi'+F)}$$

are only known to have the same maximum dimension as J'_μ, $\vec{D}'$ respectively [7].

The determination of $\mathcal{L}_4$ by the requirement that Ward identities be fulfilled is thus in general incomplete at this stage, since only the dimension of $\vec{D}$ goes unchanged through the change of effective Lagrangian, except in the case of a linear symmetry breaking. In the case of a non linear symmetry breaking, the structure of $\vec{D}$ is not arbitrary. Together with a term of given dimension and covariance under G, it must contain all terms corresponding to the ambiguities compatible with covariance and power counting [7]. The question then arises to recover in the $\mathcal{L}_4$ version the information which characterizes the covariance of the breaking.

It is conjectured that this information can be recovered by studying the high momentum behaviour of two point and three point Green's functions via e.g. the Callan Symanzik equations.

Alternatively, let $\mathcal{L}_i$ be an irreducible tensorial term in $\mathcal{L}_{break}$, $\frac{\delta \mathcal{L}_i}{\delta \omega_\mu} = \mathcal{L}_i^\mu = \tau_{ij}^\alpha \mathcal{L}_j$ for some representation $\{\tau\}$ of $\mathcal{G}$.
Then, the following Ward identity holds:

$$\partial^\mu \langle T J_\mu^{Noether,\beta}(x) \mathcal{L}_i^\alpha(y) X \rangle = \langle T \frac{\delta \mathcal{L}_{break}}{\delta \omega_\beta}(x) \mathcal{L}_i^\alpha(y) X \rangle$$
$$+ \delta(x-y) \epsilon^{\beta\alpha}{}_\gamma \langle T \mathcal{L}_i^\gamma(y) X \rangle + \delta(x-y) \tau_{ij}^\beta \langle T \mathcal{L}_j^\alpha(y) X \rangle$$
$$+ \sum_{k \in X} \delta(x-x_k) \theta_{kj}^\beta \langle T \mathcal{L}_i^\alpha(y) X \rangle$$

One may thus look for D^α in the form $D^\alpha = \sum D_i^\alpha$ if $\mathcal{L}_{break}$ was of the form $\sum \mathcal{L}_i$
and determine the covariant operators D_i^α characterized by their maximum dimension and the fulfillment of Ward identities of the type[8]

$$\partial^\mu \langle T J_\mu^\beta(x) D_i^\alpha(y) X' \rangle = \langle T D^\beta(x) D_i^\alpha(y) X' \rangle$$
$$+ \delta(x-y) \epsilon^{\beta\alpha}{}_\gamma \langle T D_i^\gamma(y) X' \rangle + \delta(x-y) \tau_{ij}^\beta \langle T D_j^\alpha X' \rangle$$
$$+ \sum_{k \in X} \delta(x-x_k) \theta_k^\beta \langle T D_i^\alpha(y) X' \rangle$$
$$+ \sum_{k \in X} \delta(x-x_k) (\theta^\beta F)_k \langle T D_i^\alpha(y) X'_{\wedge k} \rangle$$

where the time ordered products involving pairs of composite operators may eventually not be the conventional ones.

Once the characteristics of the initial Lagrangian have been recovered, through highly non linear, recursively soluble relations between the coefficients of $\mathcal{L}_4$ (values of vertex functions at zero momentum) two kinds of situations may occur. If enough parameters are left undetermined so that all physical masses of stable particles can be chosen as free parameters, one has a theory in a physical Fock space $\mathcal{F}_{phys}$ the perturbation parameter being $\hbar$, which counts the number of loops in Feynman diagrams. If not, no perturbative treatment known at present can describe the situation in terms of the correct physical Fock space.

The algebraic complexity of the general situation looks at the moment forbidding, and the more symmetric treatment indicated in Chapter IV is by far more attractive since there all operators retain all the symmetric aspects which are completely hidden in the $\mathcal{L}_4$ formalism. The only reason, other than computational, which calls for a study of the $\mathcal{L}_4$ formalism is the present lack of treatment in the present framework of systems for which, because of the group structure, some symmetric mass parameters vanish. This is admittedly a weakness of the formalism and calls for a more complete study of the interrelations between the various solutions to the decomposition problems posed by the causality requirement. In these more difficult cases, one may either first suitably approximate the theory by one in which no vanishing mass parameter is involved and study some zero mass limit. This is the method examplified in Chapter IV. Alternatively one may modify the starting point as will be shown in Chapter V and directly define the desired theory.

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IV. Gauge Invariant Quantization of the Goldstone and Higgs Model

The general formulation of the previous sections will now be applied to the examples of the Goldstone and the Higgs model. We begin with the discussion of the Goldstone model. B. Lee and K. Symanzik developed two alternative methods of quantizing the Goldstone model[1,2] . In B. Lee's work the model is regularized and quantized in a gauge invariant manner. It is then shown that the regularization can be removed for the renormalized Feynman amplitudes. In the treatment that follows we use B. Lee's method of gauge invariant quantization but without introducing a regularization. Instead we will deal directly with the unregularized, but properly renormalized, Feynman amplitudes. It will be shown that the desired properties of the model follow easily as an application of the general theorems given in Section II and III. The connection to Symanzik's method will be discussed later.

As classical Lagrangian we propose

$$\mathcal{L}_{cl.}(s) = \partial_\mu \varphi^* \partial^\mu \varphi - (\eta_0^2 - s^2 w^2)\varphi^*\varphi - h_0(\varphi^*\varphi)^2 - \frac{\delta_0(s)}{\sqrt{2}}(\varphi+\varphi^*) \tag{IV.1}$$

with

$$\varphi = \frac{1}{\sqrt{2}}\left(\tau_1 + s\,v_0 + i\,\tau_2\right) \tag{IV.2}$$

$$\tau_1 = Z_2^{1/2}\psi\,,\quad \tau_2 = Z_2^{1/2}\chi\,,\quad v_0 = Z_2^{1/2}v$$

ψ and χ denote the properly normalized fields. Their vacuum expectation values are required to vanish

$$\langle\psi\rangle = \langle\chi\rangle = 0 \tag{IV.3}$$

while it is assumed that

$$v_0 \neq 0 \tag{IV.4}$$

Expressed in terms of the fields ψ and χ (IV.1) is to be interpreted as an s-dependent Lagrangian in the sense of Chapter II. We recall that the parameter s only serves to specify the subtraction procedure and is set equal to one finally.

$\delta_0(s)$ is a polynomial in s with

$$\delta_0(0) = 0$$

of which only the value

$$\delta_0(1) = \delta_0$$

at $s = 1$ is relevant.

The parameter ω is restricted to a permissable range which will be given later. It can be shown that the Green's functions of the theory do not depend on the value of ω [4].

For $\delta_0 = 0$ (IV.1) is the Lagrangian of the Goldstone model. Formally it is gauge invariant, but the gauge symmetry is spontaneously broken due to (IV.4). The model describes two particles σ and π which are associated with the fields ψ or χ resp. The π-particle has zero mass and represents the Goldstone particle. The σ-particle is unstable since it can decay into π-particles by the interaction term $\psi\chi^2$ present in the Lagrangian (IV.1).

For $\delta_0 \neq 0$ the gauge symmetry is explicitly broken by the term proportional to $\varphi + \varphi^*$. In this case the model is called the explicitly broken Goldstone model. It describes two massive particles σ and π of mass M and μ associated with the fields ψ or χ. We always assume $\mu < M$. The σ-particle is stable only for $M < 2\mu$. In the unstable case M denotes an appropriate mass parameter related to the complex pole of the ψ-propagator [5]. In the Goldstone limit $\delta_0 \to 0$ the π-particle becomes massless, i.e. $\mu \to 0$.

Unfortunately, the Lagrangian (IV.1) is not meaningful for $\delta_0 = 0$ since some of its coefficients are infrared divergent. In order to bypass infrared problems the case $\delta_0 \neq 0$, or $\mu \neq 0$, is considered first. Then the perturbation expansion is well-defined with finite coefficients of the Lagrangian (IV.1). Eventually the Goldstone limit $\mu \to 0$ is applied yielding Green's functions and S-matrix of the Goldstone model.

Apart from s and ω the independent parameters of the theory will be the σ-mass M (or an appropriate mass parameter M in the unstable case), the π-mass μ and a suitably defined coupling constant g. Five renormalization conditions are required to hold at $s = 1$ [6]. These conditions uniquely determine the parameters η_0, h_0, δ_0, v_0, z_0 as power series in g with coefficients depending on M, μ and ω^2, but independent of s.

We now determine the free part of the Lagrangian (IV.1) following the instructions of Chapter II. Let $\mathcal{L}_{eff\,Q}$ denote the sum of all terms in $\mathcal{L}_{eff}$ which are quadratic in ψ and χ or their derivatives.

$$\mathcal{L}_{eff\,Q} = \frac{1}{2} z_2 (\partial_\mu \psi \partial^\mu \psi) + \frac{1}{2} z_2 (\partial_\mu \chi \partial^\mu \chi)$$
$$- \frac{1}{2} (\eta_0^2 - s^2\omega^2 + 3 h_0 s^2 v_0^2) \psi^2$$
$$- \frac{1}{2} (\eta_0^2 - s^2\omega^2 + h_0 s^2 v_0^2) \chi^2 \qquad \text{(IV.5)}$$

The free part of the effective Lagrangian is defined by (IV.5) with the coefficients replaced by their zero order values:

$$\mathcal{L}_{eff\,0} = \frac{1}{2}\partial_\mu\psi\,\partial^\mu\psi + \frac{1}{2}\partial_\mu\chi\,\partial^\mu\chi - \frac{1}{2}M^2(s)\,\psi^2 - \frac{1}{2}\mu^2(s)\,\chi^2 , \tag{IV.6}$$

$$\lim_{g\to 0} Z_2 = 1,$$

$$M^2(s) = \lim_{g\to 0}\left(\eta_0^2 - s^2 w^2 + 3 h_0 s^2 v_0^2\right), \tag{IV.7}$$

$$\mu^2(s) = \lim_{g\to 0}\left(\eta_0^2 - s^2 w^2 + h_0 s^2 v_0^2\right) .$$

The mass parameters M^2, μ^2 equal the values of (IV.7) at $s = 1$

$$M^2 = M^2(1) \quad , \quad \mu^2 = \mu^2(1)$$

Since η_0, w, h_0, v_0 are independent of s we obtain

$$\lim_{g\to 0} h_0 v_0^2 = \frac{M^2 - \mu^2}{2} \tag{IV.8}$$

$$M^2(s) = \frac{3}{2}\mu^2 - \frac{1}{2}M^2 + w_0^2 + s^2\left(\frac{3}{2}(M^2-\mu^2) - w_0^2\right)$$

$$\mu^2(s) = \frac{3}{2}\mu^2 - \frac{1}{2}M^2 + w_0^2 + s^2\left(\frac{1}{2}(M^2-\mu^2) - w_0^2\right) \tag{IV.9}$$

w_0 denotes the zero order value of w.

Since we need the theory in the range $0 \leq s \leq 1$ we impose the consistency condition

$$M^2(s) \geq 0 \qquad \mu^2(s) \geq 0$$

As long as $M > 0$, $\mu > 0$ the model is not expected to suffer from infrared problems. One should therefore avoid vanishing mass values in the range $0 \leq s \leq 1$ by imposing the stronger condition

$$M^2(s) > 0 \,, \quad \mu^2(s) > 0 \qquad \text{if} \quad M^2 > 0, \ \mu^2 > 0. \tag{IV.10}$$

This restricts the permissable values of ω by

$$\omega_0^2 > \frac{1}{2} M^2 - \frac{3}{2} \mu^2 \tag{IV.11}$$

Particular convenient is a choice of ω for which

$$\omega_0^2 = \lim_{g \to 0} k_0 v_0^2 = \frac{1}{2}\left(M^2 - \mu^2\right) > \frac{1}{2} M^2 - \frac{3}{2} \mu^2 \tag{IV.12}$$

With this the s-dependence (IV.9) of the masses becomes

$$\begin{aligned} M^2(s) &= \mu^2 + s^2 \left(M^2 - \mu^2\right) \\ \mu^2(s) &= \mu^2 \end{aligned} \tag{IV.13}$$

Hence the π-mass is independent of s while the σ-mass has the simple form (IV.13).

With the free Lagrangian (IV.6) involving the s-dependent masses (IV.7) the perturbation expansion (II.14) of the Green's functions is completely determined.

The main advantage of this gauge invariant formulation is that partial current conservation follows quite naturally, essentially by following the classical derivation. The current operator is defined by taking the minimal normal product of the classical Noether current

$$j_\mu = i N \left[\varphi \partial_\mu \varphi^* - \varphi^* \partial_\mu \varphi \right] \tag{IV.14}$$

More precisely, the symbol N means that the minimal normal product should be applied to each monomial of the current expressed in terms of ψ and χ. According to Chapter III the Green's functions of this current operator satisfy the Ward identities

$$\begin{aligned} \partial_\mu \langle T\, j^\mu(x)\, \varphi(u_1) \dots \varphi^*(v_1) \dots \rangle &= i \sum_k \delta(x - u_k) \langle T\, \varphi(u_1) \dots \varphi^*(v_k) \dots \rangle \\ - i \sum_k \delta(x - v_k) &\langle T\, \varphi(u_1) \dots \varphi^*(v_1) \dots \rangle \\ + i \frac{\delta_0(s)}{\sqrt{2}} &\langle T \left(\varphi(x) + \varphi^*(x) \right) \varphi(u_1) \dots \varphi^*(v_1) \dots \rangle \end{aligned}$$

$$\partial_\mu \langle T\, j^\mu(x)\, \varphi(u_1)\dots \varphi^*(v_1)\dots\rangle =$$
$$\sum_k \delta(x-u_k)\langle T\, \varphi(u_1)\dots \varphi^*(v_1)\dots\rangle$$
$$- \sum_k \delta(x-v_k)\langle T\, \varphi(u_1)\dots \varphi^*(v_1)\dots\rangle \qquad \text{(IV.15)}$$
$$+ i\,\frac{\delta_0(s)}{\sqrt{2}}\langle T\,(\varphi(x)-\varphi^*(x))\,\varphi(u_1)\dots \varphi^*(v_1)\dots\rangle$$

In operator form the law of partial current conservation becomes

$$\partial_\mu j^\mu(x) = i\,\frac{\delta_0(s)}{\sqrt{2}}\,(\varphi(x)-\varphi^*(x)) \qquad \text{(IV.16)}$$

(IV.16) follows from (IV.15) by applying the reduction formulae to the fields Ψ and χ of arguments u_i and v_i.

It can be shown that in the Goldstone limit $\mu \to 0$ the time-ordered Green's function and the S-matrix exist [7]. The current of the Goldstone model is conserved.

It is characteristic for the gauge invariant approach that - apart from the linear term - the fields Ψ and χ only appear in the gauge invariant combinations of the Lagrangian (IV.1). Without destroying the gauge invariant form of the non-linear part the Lagrangian (IV.1) may be replaced by an s-independent Lagrangian of type (II.43-45). As was discussed in Chapter II such Lagrangians suffer from a summation problem in finite order of perturbation theory. In the present case the Lagrangian contains terms of the form

$$N_4\,\Psi^2 - N_2\,\Psi^2\,, \quad N_4\,\chi^2 - N_2(\chi^2) \qquad \text{(IV.17)}$$

where the coefficients do not vanish in zero order [8]. Thus an infinite number of Feynman diagrams appears in any given order of perturbation theory.

On the other hand the equivalence theorem (equ. (II.46-47)) allows to construct an equivalent Lagrangian which contains only N_4-products and does not involve a summation problem in finite order. The Lagrangian is of the form

$$\mathcal{L}_{eff} = \sum_{j=1}^{9} C_j\, N_4(M_j) \qquad \text{(IV.18)}$$

where the M_j denote the monomials

$$\psi^2, \psi^3, \psi^4, \chi^2, \chi^4, \varphi\chi^2, \varphi^2\chi^2, \partial_\mu\varphi\,\partial^\mu\varphi, \partial_\mu\chi\,\partial^\mu\chi$$

The perturbation expansion based on (IV.18) represents Symanzik's method of renormalizing the explicitly broken Goldstone model [3]. While the Lagrangian is not manifestly gauge invariant, the coefficients of the coupling terms are correlated in such a way that the Ward identities hold in the desired form.

The renormalization of the Higgs model was first developed by B. Lee by applying Symanzik's method to a regularized version [9]. B. Lee and Zinn-Justin extended the method of gauge invariant quantization to Higgs-Kibble models including the non-Abelian case [10]. In the remainder of this chapter we use the approach of gauge invariant quantization to renormalize the Abelian Higgs model by applying the general methods of Chapter II and III. As s-dependent Lagrangian we propose

$$\begin{aligned}\mathcal{L}_d = {} & (D_\mu\varphi)^*(D^\mu\varphi) - (\eta_0^2 - s^2 w^2)\varphi^*\varphi - h_0(\varphi^*\varphi)^2 \\ & - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \frac{1}{2} m_0^2 B_\mu B^\mu - \frac{1}{2\alpha_0}(\partial_\mu B^\mu)^2 \\ & + \frac{\delta_0(s)}{\sqrt{2}}(\varphi+\varphi^*)\end{aligned} \tag{IV.19}$$

with

$$D_\mu = \partial_\mu - i e_0 B_\mu \; ; \quad B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

$$B_\mu = Z_3^{1/2} A_\mu \, , \; e_0 = Z_3^{-\frac{1}{2}} e \, , \; \alpha_0 = Z_3 \alpha \, , \; m_0 = Z_3^{-\frac{1}{2}} m \tag{IV.20}$$

$$\varphi = \frac{1}{\sqrt{2}}(\tau_1 + s v_0 + i\tau_2)$$

$$\tau_1 = Z_2^{1/2}\psi \, , \; \tau_2 = Z_2^{1/2}\chi \, , \; v_0 = Z_2^{1/2} v$$

ψ, χ and A_μ denote the properly normalized fields. Their vacuum expectation values are required to vanish while it is assumed that

$$V_0 \neq 0 \tag{IV.21}$$

The gauge class used in (IV.19) is the analogue of the Gupta-Bleuler or Stueckelberg gauge in (massive) electrodynamics. The treatment of the Higgs model in the 't Hooft gauge will be discussed in Chapter V.

For $\delta_0 = 0$ and $m = 0$ (IV.19) is the Lagrangian of the Higgs model. The model described by (IV.19) for $\delta_0 = 0$, but non-vanishing original photon mass $m \neq 0$ will be called the pre-Higgs model. In both cases the Lagrangian (IV.19) is formally gauge invariant, but the gauge symmetry is spontaneously broken due to (IV.21).

For $\delta_0 \neq 0$ the gauge symmetry is explicitly broken by the term proportional to $\varphi + \varphi^*$. In this case the model is called the explicitly broken pre-Higgs model. No infrared problems occur for $\delta_0 \neq 0$ and the Lagrangian (IV.19) can be used for setting up renormalized perturbation theory without difficulties. For $\delta_0 = 0$ infrared divergencies occur for some coefficients of (IV.19), but the Goldstone limit $\delta_0 \to 0$ and the subsequent Higgs limit $m \to 0$ (in the Landau gauge) may be applied yielding the Green's functions of the pre-Higgs and the Higgs model.

The free Lagrangian of the explicitly broken Higgs model is

$$\begin{aligned} \mathcal{L}_{eff\,0} = & -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_t^2(s) A_\mu A^\mu - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 \\ & + \frac{1}{2} \left(\partial_\mu \psi \, \partial^\mu \psi + \partial_\mu \chi \, \partial^\mu \chi \right) + w(s) A_\mu \partial^\mu \chi \\ & - \frac{1}{2} M^2(s) \psi^2 - \frac{1}{2} \mu^2(s) \chi^2 . \end{aligned} \tag{IV.22}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

The s-dependence of the coefficients becomes (IV.9) and

$$\begin{aligned} m_t^2(s) &= m^2 + s^2 w^2 \\ w(s) &= s\, w \end{aligned} \tag{IV.23}$$

with

$$\begin{aligned} v_{(0)} &= \lim_{e \to 0} v \,, \qquad w = \lim_{e \to 0} e v \\ \lim_{e \to 0} h_0 v^2 &= \frac{M^2 - \mu^2}{2} \end{aligned} \tag{IV.24}$$

For $M, m, \mu, w > 0$ the free theory can be shown to be consistent with positive masses in the range $0 \leq s \leq 1$ if μ is sufficiently small and v chosen to satisfy (IV.11).

The particles of the theory are determined from the free Lagrangian (IV.22) at $s = 1$. Following is a table of the particles, their masses and associated fields.

Particle	Mass [11]	Field
Vector meson	m_t	A_μ
ζ'	M	ψ
π	$\varkappa$	$\partial_\mu A^\mu, \chi$
Ghost particle	λ	

with

$$m_t^2 = m^2 + w^2$$

$$\varkappa^2 = \frac{1}{2}\left(\alpha m^2 + \mu^2\right) - \frac{1}{2}\sqrt{\left(\alpha m_t^2 + \mu^2\right) - 4\alpha\mu^2 m_t^2} \tag{IV.25}$$

$$\lambda^2 = \frac{1}{2}\left(\alpha m^2 + \mu^2\right) + \frac{1}{2}\sqrt{\left(\alpha m_t^2 + \mu^2\right) - 4\alpha\mu^2 m_t^2}$$

In the Goldstone limit $\delta_0 \to 0$, or equivalently $\mu \to 0$, the π-particle becomes massless ($\varkappa \to 0$) and represents the Goldstone particle of the massive Higgs model.

As usual an indefinite metric formulation is employed in order to quantize the Lagrangian (IV.19). In general, the S-matrix will not be unitary since the ghost particles of negative probabilities participate in the interaction. No physical interpretation of the model is possible then. In the Goldstone limit, however, the ghost particles are expected to decouple from the rest of the system. The argument proceeds as in electrodynamics. The ghost particles are described by the divergence $\partial_\mu A^\mu$ of the vector potential. The field equation (II.29) of the vector potential reads

$$-\partial_\nu F^{\mu\nu} + \frac{1}{\alpha}\partial^\mu \partial_\nu A^\nu + m_t^2 A^\mu = j^\mu \tag{IV.26}$$

with the current operator

$$j^\mu = ie N\left[\varphi (D^\mu \varphi)^* - \varphi^* D^\mu \varphi\right] + (Z_3 - 1)\partial_\nu F^{\mu\nu} \tag{IV.27}$$

The current operator is partially conserved

$$\partial_\mu j^\mu = \frac{1}{\sqrt{2}}\delta e\,(\varphi - \varphi^*) = \delta e\, Z_2^{-1/2}\chi \tag{IV.28}$$

as follows from Ward identities similar to (IV.15). (IV.26) and (IV.28) yield

$$(IV.29)\quad (\Box+\alpha m^2)\,\partial_\mu A^\mu = i\alpha\,\delta e\, Z_2^{1/2}\,\chi$$

as field equation of $\partial_\mu A^\mu$. In the Goldstone limit $\delta_e \to 0$ the divergence $\partial_\mu A^\mu$ becomes a free field

$$(IV.30)\quad (\Box+\alpha m^2)\,\partial_\mu A^\mu = 0$$

and the ghost particles decouple. Accordingly the S-matrix of the massive Higgs model is unitary. By using differential equations of the Callan-Symanzik type it can also be shown that the Green's functions are well defined in the Goldstone limit $\mu \to 0$ [12].

The only stable particles of the massive Higgs model are the π -particle and the free ghost particle.

In the limit $m \to 0$, the pre-Higgs model passes over into the Higgs model. The zero-mass π-particles decouple, with the massive spin-one unstable particles of the pre-Higgs model becoming stable in the limit. The massive spin-zero particles also become stable, provided $M < 2w$. The Green's functions of the A_μ, ψ and χ fields can be shown to exist in the Higgs limit, but only in Landau gauge ($\alpha = 0$) [12].

The equivalence theorem (II.46 - 47) can be applied to construct an equivalent Lagrangian consisting of N_4-products only. In contradistinction to (IV.19) the non-linear part of the N_4-Lagrangian is not manifestly gauge invariant. This N_4-version of the Higgs model represents B. Lee's original approach in the language of the normal product formalism.

Chapter IV: References and Footnotes

[1] The material of this section will be published in a series of papers by J.H. Lowenstein, M. Weinstein, W. Zimmermann (part I and II), and J.H. Lowenstein, B. Schroer (part III).

[2] B. Lee, Nucl. Phys. B9, 649 (1969).

[3] K. Symanzik, Lett. Nuovo Cimento 2, 10 (1969) and Commun. Math. Phys. 16, 48 (1970).

[4] For the proof see part II of ref. [1].

[5] For problems concerning unstable particles in perturbation theory we refer to M. Veltman, Physica 29, 122 (1969) and part III of ref. [1].

[6] For the formulation of the renormalization conditions see ref. [1].

[7] K. Symanzik, Lett. Nuovo Cimento 2, 10 (1969) and Commun. Math. Phys. 16, 48 (1970).
F. Jegerlehner and B. Schroer, to be published.

[8] By appropriate choice of $\mathfrak{W}$ one of the coefficients can be made to vanish in zero order, but not both.

[9] B. Lee, Phys. Rev. D5, 823 (1972).

[10] B. Lee and J. Zinn-Justin, to be published.

[11] The values given are the masses in zero order. Only for stable particles may the zero order values be identified with the exact masses by suitable normalization conditions.

[12] See ref. [1], part III.

V. Models with vanishing Symmetric Mass Parameters

As stressed in chapter III, the combinatorics of intermediate renormalization comes into conflict with the possible occurrence of vanishing mass parameters, and, in particular, another description of symmetry breaking has to be found if the group structure implies the vanishing of some mass parameters. Typical examples of such symmetries are chiral symmetries, when spin 1/2 fields are involved, and gauge symmetries, although no difficulty should in principle arise if the physical masses are to be non vanishing. In such cases, one may first consider the classical theory as a limit of a theory where no vanishing mass parameter occurs, quantize the latter and let the spurious mass parameters go to zero. This is the road chosen in Chapter IV. The only alternative strategy which is known at present [1] is to investigate the structure of the classical Lagrangian which describes the tree approximation of the theory to be constructed, characterize the symmetry via the Ward identities which express the classical Noether theorem in presence of external sources, and look for an L_4 Lagrangian for which Ward identities of the type found in the tree approximation hold in finite renormalized form, in the sense that no composite operator different from those found in the tree approximation occur, with, however, possibly different coefficients. In case such a program cannot be completed, with composite operators occurring with their naive dimension, renormalized Ward identities are said to contain anomalies - e.g. of the type found in the trace identity for the energy momentum tensor cf. Ch. III. There exists at the moment, unfortunately, no general theorem which allows to predict from the group structure the presence of anomalies.

Since the present program is still at a very experimental stage, we shall content ourselves with the description of two examples: the σ model with nucleons and the Abelian Higgs Kibble model treated in 't Hooft's gauge.

1. The σ model involving nucleons [2]

Choosing for simplicity the chiral group to be $U(1)\times U(1)$, and denoting the meson field (π,σ), the nucleon field, ψ, the Lagrangian in the tree approximation is obtained from the formal Lagrangian

$$\mathcal{L}_{\text{formal}} = \bar{\psi}\gamma\frac{1}{i}\partial\psi + g\bar{\psi}(\sigma + i\pi\gamma_5\psi) + \frac{1}{2}\left(\partial_\mu\pi\,\partial^\mu\pi + \partial_\mu\sigma\,\partial^\mu\sigma\right)$$
$$- \frac{m^2}{2}\left(\pi^2+\sigma^2\right) - \frac{\lambda}{4}\left(\pi^2+\sigma^2\right)^2 + C\sigma$$

by the field translation $\sigma = \sigma' + F$, under the constraint that no term linear in σ' remains.
We thus obtain

$$\mathcal{L}_{tree} = \bar{\psi}\gamma\tfrac{1}{i}\partial\psi + gF\bar{\psi}\psi + g\bar{\psi}(\sigma' + i\pi\gamma_5)\psi$$
$$+ \tfrac{1}{2}(\partial_\mu\pi\partial^\mu\pi + \partial_\mu\sigma'\partial^\mu\sigma') - \frac{m^2+\lambda F^2}{2}\pi^2 - \frac{m^2+3\lambda F^2}{2}\sigma'^2$$
$$- \frac{\lambda}{4}(\pi^2+\sigma'^2)^2 - \lambda F\sigma(\sigma^2+\pi^2)$$

with the constraint $C = \lambda F^3 + m^2 F$
This Lagrangian may be written as follows

$$\mathcal{L}_{tree} = \bar{\psi}\left(\gamma\tfrac{1}{i}\partial + M\right)\psi + g\bar{\psi}(\sigma' + i\pi\gamma_5)\psi$$
$$+ \tfrac{1}{2}\partial_\mu\pi\partial^\mu\pi - \frac{m_\pi^2}{2}\pi^2 + \tfrac{1}{2}\partial_\mu\sigma'\partial^\mu\sigma' - \frac{m_\sigma^2}{2}\sigma'^2$$
$$- g^2\frac{m_\sigma^2 - m_\pi^2}{8M^2}(\pi^2+\sigma'^2)^2 - g\frac{m_\sigma^2 - m_\pi^2}{2M}\sigma'(\pi^2+\sigma'^2)$$

where the following change of parameters was made:

$$M = gF$$
$$m^2 + \lambda F^2 = m_\pi^2$$
$$m^2 + 3\lambda F^2 = m_\sigma^2$$

which implies $\lambda = g^2\frac{m_\sigma^2 - m_\pi^2}{2M^2}$

(Instead of M, m_π, m_σ, g we could have chosen as parameters M, m_π, m_σ, F).

The Ward identities are most easily obtained by applying Noether's theorem to $\mathcal{L}_{formal}$ + source term and performing the field translation afterwards. The result is

$$\partial^\mu \langle T J_\mu(x)\, X\rangle^{tree} = \sum_{k\in X}\delta(x-x_k)\,\theta_k\langle T\, X\rangle$$
$$+ \sum_{\pi\in X}\delta(x-x_\pi)\,F\langle T\, X_{\Delta_\pi}\rangle$$
$$+ C\langle T\,\pi(x)\,X\rangle$$

where θ_μ represents an infinitesimal chiral transformation on field (k) ($\pi \to \sigma$, $\sigma \to -\pi$, $\psi \to i\gamma_5 \psi$ $\bar\psi \to -\bar\psi i\gamma_5$) and

One now looks for an effective Lagrangian

$$\begin{aligned}\mathcal{L}_4 = {} & (1+B)\,\bar\psi\,\gamma\,\tfrac{1}{i}\partial\,\psi + (M+A)\,\bar\psi\,\psi + g_\sigma(1+h_\sigma)\,\bar\psi\sigma\psi \\ & + g(1+h_\pi)\,\bar\psi\, i\pi\gamma_5\psi + \tfrac{1}{2}(1+b_\pi)\,\partial_\mu\pi\,\partial^\mu\pi - \tfrac{1}{2}(m_\pi^2+a_\pi)\pi^2 \\ & + \tfrac{1}{2}(1+b_\sigma)\,\partial_\mu\sigma\,\partial^\mu\sigma - \tfrac{1}{2}(m_\sigma^2+a_\sigma)\sigma^2 \\ & - g^2\,\frac{m_\sigma^2-m_\pi^2}{8M^2}\left[(1+\ell_\pi)\pi^4 + 2(1+\ell_{\pi\sigma})\pi^2\sigma^2 + (1+\ell_\sigma)\sigma^4\right] \\ & - g\,\frac{m_\sigma^2-m_\pi^2}{2M}\left[(1+k_\sigma)\sigma^3 + (1+k_\pi)\sigma\pi^2\right]\end{aligned}$$

which depends on 13 parameters, and for a current

$$\begin{aligned}J_\mu = {} & (1+\alpha_\pi)\,\pi\,\partial_\mu\sigma - (1+\alpha_\sigma)\,\sigma\,\partial_\mu\pi + F(1+\beta_\pi)\,\partial_\mu\pi \\ & + (1+\gamma)\,\bar\psi\,\gamma_\mu\gamma_5\,\psi\end{aligned}$$

such that a Ward identity identical with that obtained in the tree approximation holds. This is possible and leaves freedom, without varying F, to fix the masses of π , σ, ψ at their zeroth order values m_π, m_σ, M and to set equal to unity the residues of the propagators of the π and the ψ fields at their respective poles. The number of free parameters is just the number of parameters occurring in the most general formal Lagrangian invariant under chiral transformations, except for a linear breaking term.

2. The Abelian Higgs Kibble model in the 't Hooft gauge.[3]

The interest of this model within the general class of spontaneously broken gauge models is that it exhibits most features characteristic of these models in so far as ultraviolet behaviour is concerned. Infrared difficulties, on the other hand, are avoided.

One starts from the formal Lagrangian

$$\mathcal{L}_{formal} = -\frac{1}{4} G_{\mu\nu} G^{\mu\nu} + (D_\mu \varphi)^* (D^\mu \varphi) + \mu^2 \varphi^* \varphi$$
$$- g(\varphi^* \varphi)^2 - \frac{1}{2\alpha} (\partial_\mu A^\mu + \rho \varphi_2)^2 + \frac{1}{\alpha} \bar{c} m c$$

where, for the time being, the Faddeev-Popov term, $\bar{c} m c$, whose convenience will be seen later, is ignored. The mode corresponding to broken symmetry is obtained by making the substitution

$$\varphi = \frac{\varphi_1 + v + i\varphi_2}{\sqrt{2}}$$

v being determined by the condition that no term linear in φ_1 survives:

$$\mu^2 = g v^2$$

A translation induced photon mass term yields a transverse photon mass

$$m^2 = e^2 v^2$$

The mass corresponding to the φ_1 field is

$$M^2 = -\mu^2 + 3 g v^2 = 2 g v^2$$

The mass matrix corresponding to the coupled $\varphi_2, \partial_\mu A^\mu$ quadratic form yields a degenerate eigenvalue at

$$\lambda^2 = \rho m$$

Before the introduction of the Faddeev Popov part of the Lagrangian the only non gauge invariant part of the Lagrangian is given by the gauge term

$$-\frac{G^2}{2\alpha} = -\frac{(\partial_\mu A^\mu + \rho \varphi_2)^2}{2\alpha}$$

In terms of the new variables, the Lagrangian reads:

$$\mathcal{L}_{tree} = \frac{1}{2} \partial_\mu \varphi_1 \partial^\mu \varphi_1 - \frac{M^2}{2} \varphi_1^2 + \frac{1}{2} \partial_\mu \varphi_2 \partial^\mu \varphi_2 - \frac{\rho^2}{2\alpha} \varphi_2^2$$
$$+ \left(m - \frac{\rho}{\alpha}\right) \varphi_2 \partial_\mu A^\mu - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 - \frac{1}{4} G_{\mu\nu} G^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu$$
$$- e A_\mu (\varphi_1 \partial^\mu \varphi_2 - \varphi_2 \partial^\mu \varphi_1) + \frac{e^2}{2} A_\mu A^\mu (\varphi_1^2 + \varphi_2^2)$$
$$+ e m A_\mu A^\mu \varphi_1 - \frac{e^2 M^2}{2 m^2} (\varphi_1^2 + \varphi_2^2)^2 - \frac{2 e M^2}{m} \varphi_1 (\varphi_1^2 + \varphi_2^2)$$

where ρ was eliminated against e through the relations

$$2 g v^2 = M^2, \quad e^2 v^2 = m^2, \quad \Rightarrow \quad \frac{2g}{e^2} = \frac{M^2}{m^2}$$

The restricted 't Hooft gauge is obtained for $\rho = \alpha m$ whereby the $\varphi_2 \partial_\mu A^\mu$ cross term vanishes. The gauge invariance is best expressed by applying the integrated Noether theorem corresponding to the gauge group to $\mathcal{L}_{formal}$ + source terms, and performing the field translation on the corresponding Ward identity, which assumes the form

$$0 = -\int \frac{1}{\alpha} \frac{\delta G_x}{\delta \Lambda_y} G_x + \text{source terms} = \frac{1}{\alpha} \int m_{yx} G_x + \text{source terms}$$

On the other hand, the question of the gauge invariance of the physical scattering amplitudes first proceeds through the study of the variation of the connected Green's functional under the change of the gauge parameters γ (here, α, ρ):

$$\frac{\partial G^c(J)}{\partial \gamma} = -i \int \frac{\partial}{\partial \gamma} \left(\frac{G^2}{2\alpha} \right) = -i \int \frac{G}{\sqrt{\alpha}} \frac{\partial}{\partial \gamma} \left(\frac{G}{\sqrt{\alpha}} \right)$$

where use has been made of the fact that the Green's functional is the Legendre transform of the Lagrangian.[1],[2] Gauge invariance is achieved if

$$\langle G \rangle_{phys} = 0$$

for a suitable definition of physical states. In any event, it is desirable to convert the Ward identity into an identity of the Slavnov type:[3]

$$G_x = \cdots$$

which requires the inversion of the - in general field dependent - differential operator m. This is best achieved by introducing the Faddeev Propov fields with Lagrangian $\frac{1}{\alpha} \bar{C} m C$ and source terms $\bar{\xi} C + \bar{C} \xi$. The Ward identity then reads

$$0 = \int -\frac{1}{\alpha} m_{yx} G_x + \frac{1}{\alpha} \bar{C}_x \frac{\delta m_{xt}}{\delta \Lambda_y} C_t + \text{source terms}$$

and the Slavnov identity is obtained by integrating over y after multiplication by $\bar{C}_y$[5]! Using the equations of motion, the first term becomes

$$\int \bar{\xi}_x \frac{G}{\sigma x}$$

whereas the second term vanishes if the Faddeev Propov ghost fields obey Fermi's statistics:

$$\int dx_3\, dz_3\, dt\; \bar{C}_3\, \bar{C}_3\, \frac{\delta G_t}{\delta \Lambda_3\, \delta \Lambda_3}\, C_t = 0$$

by virtue of the abelianness of the gauge group. In the non abelian case,[3] this term does not vanish, but assumes a particularly simple form, independent of the gauge function: $\bar{C}\vec{t}\bar{C}\vec{t}$ where $\vec{t}$ denotes the infinitesimal generator of the internal Lie algebra of the gauge group, only in the case where the Faddeev Popov ghost fields obey Fermi's statistics.

Written in full, in terms of the translated field variables, the Slavnov identities read in the tree approximations

$$\langle T\,(\partial_\mu A^\mu + \rho\varphi_2)(x)\,\varphi_1(x_1)\cdots\varphi_1(x_n)\,A_{\mu_1}(y_1)\cdots A_{\mu_m}(y_m)\,\varphi_2(z_1)\cdots\varphi_2(z_p)$$
$$C(t_1)\cdots C(t_k)\;\bar{C}(u_1)\cdots\bar{C}(u_k)\rangle^c =$$

$$-\,e\sum_{i=1}^{n}\langle T\,C(x)\,\varphi_1(x_1)\cdots(\varphi_2\bar{C})(x_i)\cdots\varphi_1(x_n)\cdots\bar{C}(u_k)\rangle^c$$

$$+\sum_{i=1}^{m}\langle T\,C(x)\,\varphi_2(x_1)\cdots\varphi_2(x_n)\,A_{\mu_1}(y_1)\cdots\partial_{\mu_i}\bar{C}(y_i)\cdots\bar{C}(u_k)\rangle^c$$

$$+\,m\sum_{i=1}^{p}\langle T\,C(x)\,\varphi_2(x_1)\cdots\varphi_2(x_n)\,A_{\mu_1}(y_1)\cdots A_{\mu_m}(y_m)\,\varphi_2(z_1)\cdots\bar{C}(z_i)\cdots\bar{C}(u_k)\rangle^c$$

$$+\,e\sum_{i=1}^{p}\langle T\,C(x)\,\varphi_2(x_1)\cdots\varphi_1(x_n)\,A_{\mu_1}(y_1)\cdots A_{\mu_m}(y_m)\,\varphi_2(z_1)\cdots(\varphi_2\bar{C})(z_i)\cdots\bar{C}(u_k)\rangle^c$$

$$-\sum_{i=1}^{k}\langle T\,C(x)\cdots C(t_1)\cdots(\partial_\mu A^\mu+\rho\varphi_2)(t_i)\cdots C(t_k)\,\bar{C}(u_1)\cdots\bar{C}(u_k)\rangle^c$$

The reason for including Faddeev Popov fields within the Slavnov identities is that among other things, it is believed that they will turn out to be relevant to the unitarity problem of the physical S operator for the fully renormalized theory.

One can show that conversely, if such identities are to hold in the tree approximation the Lagrangian must be of the form initially postulated except for the possible occurrence of wave function renormalization factors Z_A, Z_ϕ in front of the terms $-\frac{1}{4} G_{\mu\nu} G^{\mu\nu}$, $(D_\mu \varphi)^* (D_\mu \varphi)$, respectively. Written in full, the Faddeev Popov contribution to $\mathcal{L}_{tree}$ is

$$\frac{1}{\alpha} \left(\partial_\mu \bar{C} \, \partial^\mu C + \rho m \, \bar{C} C + \rho e \varphi_1 \bar{C} C \right)$$

The Lagrangian then depends on the following parameters:

$$Z_A, m; Z_\phi, M, \rho, e, \alpha$$

which can be characterized as follows:

Z_A, m are related to the residue and pole of the transverse photon propagator

Z_ϕ, M are related to the residue and pole of the φ_1 field propagator

ρ is related to the common $[\partial_\mu A^\mu, \varphi_2, C]$ ghost propagator pole [4]

α is related to the residue of the Faddeev Popov ghost propagator

e is related to the $\varphi_1 \varphi_1 \varphi_1$ or $\varphi_1 \varphi_1 A^T$ scattering. Alternatively, g is related to $\varphi_1 \varphi_1 \varphi_1 \varphi_1$ scattering. In all the following, we shall assume m, M and the ghost mass to be restricted by inequalities which insure the stability of all three types of particles. In order that the physical scattering amplitudes be gauge invariant, one sees that, by virtue of the Slavnov identities, physical states should only contain φ_1, and A^T quanta.

The question is now to construct the most general dimension four effective Lagrangian whose zeroth order approximation in terms of the loop counting parameter coincides with $\mathcal{L}_{tree}$ and inquire whether constraints can be put on the coefficients in such a way that Slavnov type identities may hold.

Let then

$$
\begin{aligned}
\mathcal{L}^{(4)}_{eff} = & -\frac{1}{4}(1+b) G_{\mu\nu} G^{\mu\nu} + \frac{m^2+a}{2} A_\mu A^\mu - \frac{1}{2\alpha'}(1+c)(\partial_\mu A^\mu)^2 \\
& + \frac{d}{4}(A_\mu A^\mu)^2 + \frac{1}{2}(1+B_1)\partial_\mu \varphi_1 \partial^\mu \varphi_1 - \frac{M^2+A_1}{2}\varphi_1^2 \\
& + \frac{1}{2}(1+B_2)\partial_\mu \varphi_2 \partial^\mu \varphi_2 - \frac{1}{2}\left(\frac{\rho^2}{\alpha}+A_2\right)\varphi_2^2 + \left(\frac{\rho}{\alpha} - m - \tilde{f}_2\right) A_\mu \partial^\mu \varphi_2 \\
& + e(1+f_1) A_\mu \varphi_2 \partial^\mu \varphi_1 - e(1+f_2) A_\mu \varphi_1 \partial^\mu \varphi_2 + \frac{e^2}{2}(1+g_1) A_\mu A^\mu \varphi_1^2 \\
& + \frac{e^2}{2}(1+g_2) A_\mu A^\mu \varphi_2^2 + em(1+\tilde{g}_1) A_\mu A^\mu \varphi_1 \\
& - \frac{e^2 M^2}{2m^2}(1+h_1)\varphi_1^4 - \frac{e^2M^2}{2m^2}(1+h_2)\varphi_2^4 - \frac{e^2M^2}{m^2}(1+h_{12})\varphi_1^2\varphi_2^2 \\
& - \frac{2eM^2}{m}(1+k_1)\varphi_1^3 - \frac{2eM^2}{m}(1+k_2)\varphi_2^2\varphi_1 \\
& + \frac{1+H}{\alpha}\Big[\partial_\mu \bar{C}\,\partial^\mu C - (\rho m + A)\bar{C}C - \rho e(1+l)\bar{C}\varphi_1 C \\
& \qquad + r_1 \bar{C}\varphi_1^2 C + r_2 \bar{C}\varphi_2^2 C + s\,\bar{C} A_\mu A^\mu C\Big]
\end{aligned}
$$

We now look for an identity of the Slavnov type, recalling the conservation of the number of Faddeev Popov ghost pairs, and the invariance under charge conjugation:

$$A_\mu \to -A_\mu\ ;\ \varphi_2 \to -\varphi_2\ ;\ \varphi_1 \to \varphi_1\ ;\ C \to C\ ;\ \bar{C} \to \bar{C}\ .$$

One can use the effective equations of motion for $\bar{C}$, $\partial_\mu A^\mu$, φ_1, φ_2 under the following combinations:[3]

$$
\begin{aligned}
\int N_S\, \bar{C}\,\Box\,\partial_\mu A^\mu &= cst \int \delta_{\bar{C}}\,\partial_\mu A^\mu + \int N_S\, \delta_{\bar{C}}\,\partial_\mu A^\mu \\
&= cst \int \bar{C}\,\partial_\mu\,\delta_{A_\mu} + \int \bar{C}\,\partial_\mu J^\mu
\end{aligned}
$$

$$\int N_4 \bar{c} \Box \varphi_2 = cst \int \varphi_2 \delta_{\bar{c}} + \int N_4 \jmath_{\bar{c}} \varphi_2$$
$$= cst \int \bar{c} \delta_{\varphi_2} + \int N_4 \bar{c} \jmath_2$$

$$\int N_5 \bar{c} \Box (\varphi_1 \varphi_2) = cst \int \delta_{\bar{c}} N_2 (\varphi_1 \varphi_2) + N_5 \jmath_{\bar{c}} \{\varphi_1 \varphi_2\}$$
$$= \int 2 \bar{c} \partial_\mu \varphi_1 \partial^\mu \varphi_2 + cst \int \bar{c} \varphi_2 \delta_{\varphi_1}$$
$$- \int N_5 \jmath_1 \{\bar{c} \varphi_2\} + cst \int \bar{c} \varphi_1 \delta_{\varphi_2} + \int N_5 \jmath_2 \{\bar{c} \varphi_1\}$$

where the brackets $\{ \cdot \}$ indicate anisotropic normal products and $\jmath_1, \jmath_2, \jmath_{\bar{c}}, \jmath_\mu$ are simply related (within the addition of mass terms and the multiplication through wave function renormalization coefficients) to the corresponding field sources.

The reduction of anisotropic to isotropic normal product, as well as the anisotropic use of the equations of motion for φ_1, φ_2 in the evaluation of $\partial^\mu \jmath_\mu$ allows to cast these three identities into the form

$$\int \partial_\mu A^\mu \delta_{\bar{c}} = \text{wanted terms + unwanted terms}$$
$$\int \varphi_2 \delta_{\bar{c}} = \text{wanted terms + unwanted terms}$$
$$\int N_2 \varphi_1 \varphi_2 \delta_{\bar{c}} = \text{wanted terms + unwanted terms}$$

Wanted terms are of the form $\int \bar{c} \partial_\mu \delta_{A_\mu}$, $\int \bar{c} \delta_{\varphi_2}$, $\int N_2 \bar{c} \varphi_1 \delta_{\varphi_2}$, $\int N_2 \bar{c} \varphi_2 \delta_{\varphi_1}$

After repeated use of Zimmermann's identities unwanted terms can be cast into the form of itegrals of the following monomials, all counted with dimension 5 :

$\bar{c} \varphi_2$,
$\bar{c} \partial_\mu A^\mu$, $\bar{c} \varphi_1 \varphi_2$

$$\bar{c}\varphi_2\varphi_1^2,\ \bar{c}\varphi_2 A_\mu A^\mu,\ \bar{c}\varphi_2^3,\ \bar{c}\varphi_1\partial_\mu A^\mu,\ \bar{c}A_\mu\partial^\mu\varphi_1,$$
$$\bar{c}\varphi_2\varphi_1^3,\ \bar{c}\varphi_2^3\varphi_1,\ \bar{c}A_\mu A^\mu\varphi_1\varphi_2,\ \bar{c}\partial_\mu A^\mu\varphi_1^2,\ \bar{c}A_\mu\partial^\mu\varphi_1\varphi_1,$$
$$\bar{c}\partial_\mu A^\mu\varphi_2^2,\ \bar{c}\varphi_2\partial_\mu\varphi_2 A^\mu,\ \bar{c}\partial_\mu A^\mu A_\lambda A^\lambda,\ \bar{c}A_\lambda\partial^\lambda(A_\mu A^\mu),$$
$$\bar{c}\partial_\mu\varphi_1\partial^\mu\varphi_2,\ \bar{c}\partial_\mu\bar{c}A^\mu c\ .$$

There are 19 unwanted terms whereas the Lagrangian depends on 25 coefficients. One of which is connected with the normalization of the Faddeev Popov ghost field. It turns out that one can express the unwanted term $N_\delta\bar{c}\varphi_1\varphi_{12}$ in terms of $\int\varphi_2\delta_{\bar{c}}$, wanted terms, and other unwanted terms, and, similarly $\int N_\delta\bar{c}\partial_\mu\varphi_1\partial^\mu\varphi_2$ in terms of $\int N_2\varphi_1\varphi_2\delta_{\bar{c}}$, wanted terms and other unwanted terms. Imposing then the vanishing of the remaining 17 unwanted terms, one gets Slavnov identities of the form:

$$\langle T\left[\partial_\mu A^\mu+(\rho+\delta\rho)\varphi_2+\delta\rho_{12}N_2(\varphi_1\varphi_2)\right]\varphi_1(x_1)\cdots\varphi_1(x_n)$$
$$A_{\mu_1}(y_1)\cdots A_{\mu_m}(y_m)\varphi_2(z_1)\cdots\varphi_2(z_p)c(t_1)\cdots c(t_k)\bar{c}(u_1)\cdots\bar{c}(u_k)\rangle^c$$
$$=-\epsilon\left(\frac{1+\beta_1}{1+H}\right)\sum_{i=1}^{n}\langle T\,c(x)\varphi_1(x_1)\cdots(\varphi_2\bar{c})(x_i)\cdots\varphi_1(x_n)\cdots\bar{c}(u_k)\rangle^c$$
$$+\left(\frac{1+\beta_A}{1+H}\right)\sum_{i=1}^{m}\langle T\,c(x)\cdots A_{\mu_i}(y_i)\cdots\partial_{\mu_i}\bar{c}(y_i)\cdots\bar{c}(u_k)\rangle^c$$
$$+m\left(\frac{1+\beta_{12}}{1+H}\right)\sum_{i=1}^{p}\langle T\,c(x)\cdots\varphi_2(z_1)\cdots\bar{c}(z_i)\cdots\varphi_2(z_p)\cdots\bar{c}(u_k)\rangle^c$$
$$+\epsilon\left(\frac{1+\beta_2}{1+H}\right)\sum_{i=1}^{p}\langle T\,c(x)\cdots\varphi_2(z_1)\cdots(\varphi_1\bar{c})(z_i)\cdots\varphi_2(z_p)\cdots\bar{c}(u_k)\rangle^c$$
$$-\sum_{i=1}^{k}\langle T\,c(x)\cdots c(t_1)\cdots\left[\partial_\mu A^\mu+(\rho+\delta\rho)\varphi_2+\delta\rho_{12}N_2(\varphi_1\varphi_2)\right](t_i)\cdots$$
$$c(t_k)\cdots\bar{c}(u_k)\rangle^c$$

where the coefficients $\delta\rho$, $\delta\rho_{12}$, β_1, β_A, β_{12}, β_2 are finite formal power series in $\hbar$. Apart from the fact that α only occurs in the combination $\alpha/(1+H)$ it is not known at present whether simple relations automatically hold between these coefficients because of the complicated way in which they were obtained. In particular, there is not a priori reason why $\delta\rho_{12}$ should vanish. In other words, had we started from a gauge function of the form $B = \partial_\mu A^\mu + \rho\varphi_2 + \rho_{12}\varphi_1\varphi_{12}$ in the tree approximation- the most general expression of dimension 2 odd under charge conjugation - we would have obtained a Slavnov identity involving a renormalized gauge function of the form $B^{Ren.} = \partial_\mu A^\mu + (\rho+\delta\rho)\varphi_2 + (\rho_{12}+\delta\rho_{12}) N_2\varphi_1\varphi_2$ and even when ρ_{12} vanishes the induced $\delta\rho_{12} N_2(\varphi_1\varphi_{12})$ term may survive. In fact, if we believe that the tree approximation property

$$\langle T\, B(x)\, B(y)\rangle \propto \delta(x-y)$$

survives through radiative corrections, one can see that the $N_2(\varphi_1\varphi_2)$ induced term has to be present.

Once the Slavnov identities have been obtained, there are, besides the combination $\alpha/(1+H)$, seven parameters left free which can be used to fulfill the following normalization conditions: position of the poles, residues being unity at these poles for the transverse photon and φ_1 propagator, mass shell value of the three or four φ_1 (or $\varphi_1\varphi_1 A^\mu$) vertices equal to the tree approximation value, and finally double vanishing of the inverse determinant of the coupled $\partial_\mu A^\mu$, φ_2 propagator matrix at $p^2 = \rho m^2$. [6] The last condition combined with the Slavnov identity implies that the Faddeev Popov ghost propagator has a simple pole at $p^2 = \rho m^2$, whose residue can be normalized to one by means of the parameter H, if one wishes to do so. The theory is then completely interpretable within a Fock space with indefinite metric whose structure will be reported elsewhere together with the appropriate asymptotic theory.

The crucial test for the correctness of this renormalization scheme of course relies on the check of unitarity and gauge invariance of the physical S operator which has not yet been attacked within the present framework, and, next the existence of local gauge invariant observables which leave the physical subspace invariant, in the same way as in quantum electrodynamics. [7]

Chapter V : References and Footnotes

[1] This is the strategy advocated by J. Schwinger in: J. Schwinger "Particles Sources and Fields", Addison Wesley Pub. Co., Reading, Mass. (1970).
J. Schwinger, "Particles and Sources", Gordon & Breach New York (1970) (Brandeis 1967). See also:
A. Rouet, R. Stora, Lectures given at the Universities of Geneva and Lausanne (1973).

[2] Ch. III, Refs. [1] and [8]. The determination of an L_4 Lagrangian such that Ward identities hold, performed in Ref. [7] of Ch. III requires one more constraint than is allowed by the number of parameters at disposal. That constraint, a relation between some r coefficients, can be shown to be automatically fulfilled by an argument concerning the high momentum behaviour of vertex functions. This argument is due to O. Piguet and similar to one used by him in the construction of an L_4 Lagrangian for the Higgs Kibble model with massive photons of Chapter IV .
(O. Piguet, private communication).

[3] More details can be found in A. Rouet, R. Stora, Ref. [1] , and A. Rouet, article in preparation.

[4] We wish to thank B.W. Lee for a discussion on this point.

[5] This way of obtaining the Slavnov identities can be found in : L. Quaranta, A. Rouet, R. Stora, E. Tirapegui "Spontaneously broken gauge invariance: Ward identities, Slavnov identities, gauge invariance", in "Renormalization of Yang Mills Fields and Applications to Particle Physics", CNRS Marseille, June 19-23 (1972), where however the treatment of renormalization is formal due to the a priori possible occurence of infrared difficulties which does not happen in the treatment given in this chapter. Concerning the renormalization of gauge field theories see G. 't Hooft and B.W. Lee's Lectures, in this volume, where the original work by G. 't Hooft, B.W. Lee, M. Veltman, J. Zinn-Justin, is reported.

[6] Such a normalization condition was used in a version of the renormalization of the Higgs Kibble model in Stueckelberg's gauge, by J.H. Löwenstein, M. Weinstein, W. Zimmermann.

[7] J.H. Löwenstein, B. Schröer, PR. D 6 , 1553 (1972)

C H A P I T R E V

RENORMALIZATION OF THE ABELIAN HIGGS-KIBBLE MODEL

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INTRODUCTION

The latest achievements on the renormalization of Lagrangian models involving gauge fields, mostly due to G. 't Hooft, B.W. Lee, M. Veltman, J. Zinn Justin [1], were primarily based on the use of a gauge invariant regularization procedure, the most popular of which being the so called dimensional regularization [2]. The gauge structure could thus conveniently be respected by fulfilling the so called Slavnov identities [3] through the renormalization procedure. There resulted finite Green's functions which could not however be directly given an interpretation relevant to an operator theory in some Fock space, were it be in a perturbative sense, because of the lack of the finite mass renormalizations which would have been necessary for this purpose. As will be seen here, an operator interpretation is quite convenient for any discussion involving asymptotic concepts concerning e.g. the unitarity of the S operator, the construction of gauge invariant local operators etc.

We shall treat here the simplest model involving gauge fields in which no infrared problem occurs, namely the abelian Higgs-Kibble model [4] within the class of gauges advocated by G. 't Hooft. The algebraic complications which occur in the non abelian cases are deferred to later publications.

We shall make full use of the combinatorial knowledge of renormalized perturbation theory that has been acquired through the work of W. Zimmermann [5] (effective Lagrangians normal products, Wilson expansions), J.H. Lowenstein [6] and YMP Lam [7] (renormalized action principle), which has been successfully applied in other cases (massive quantum electrodynamics [8], σ models [9] abelian Higgs Kibble model in the Stueckelberg gauge [10].

This well developed machinery, which relies on the locality and power counting properties of perturbation theory, is most effectively put to work [11] by intensive use of the implicit function theorem for formal power series through which, as we shall see, most symmetry aspects of the perturbation series can be read off on the classical Lagrangian on which the theory is based, including the possible occurence of anomalies. This possibly surprising statement will be widely illustrated in the present work and in reviews now in preparation [12].

The main reason why such a favourable situation prevails in the present case is that the model is almost entirely specified by an invariance property

even after the introduction of the necessary Faddeev-Popov ghosts[13]. Namely, at the classical level, the Lagrangian is invariant under transformations of the supergauge type[14], which we have called Slavnov transformations. In the abelian case treated here, one has however also to impose the full degeneracy of the ghost masses in order to implement spontaneous breaking. This is a particular feature of the abelian case which in a sense makes things more complicated.

Section I is thus devoted to a study of some crucial aspects of the tree approximation. The role of the invariance under Slavnov transformations and the particuliar expression of spontaneous breaking are stressed.

In Section II the model is defined to all orders of a perturbation expansion in powers of a parameter, $\hbar$, which counts the number of loops in Feynman diagrams. Namely, we show that both renormalized Slavnov identities and the normalization conditions on Green's functions which hold in the tree approximation can be fulfilled to all orders. The compactness of the proofs is due to a repeated use of the implicit function theorem for formal power series[11]. The logic of the construction also makes clear how anomalies, which do not occur in the present model, can be produced.

In Section III, one proves the independence of the physical scattering operator against a change of the parameters which label the gauge function, by suitably generalizing the argument given by J.H. Lowenstein and B. Schroer[8] in the case of massive quantum electrodynamics.

Section IV is devoted to a direct combinatorial proof of the unitarity of the physical S operator.

Several appendices are devoted to a number of technical questions:

Appendix I deals with the structure of the Slavnov identities at the classical level in the non abelian case.

Appendix II is devoted to a brief description of the implicit function theorem for formal power series[11]

Appendices III, IV and V give some computational details which would have

obscured the line of argument in the body of the article.

Appendix VI deals with the construction of some local gauge invariant operators of dimension smaller than or equal to four.

Appendix VII extends the theory to quadratic gauges odd under charge conjugation.

I - THE TREE APPROXIMATION

As is well known, a classical Lagrangian, $\mathcal{L}^{cl}$, which will be assumed to be of the renormalizable type, defines the tree approximation of a quantum Green's functional

$$\mathcal{Z}(\underline{J}) = \exp \frac{i}{\hbar} \mathcal{Z}^{c}(\underline{J}) \tag{1}$$

where $\underline{J}$ denotes collectively a set of sources linearly coupled to the field variables $\underline{\varphi}$ from which $\mathcal{L}^{cl}$ is constructed. The Legendre transform [15] $\Gamma(\underline{\varphi})$ of the connected Green's functional $\mathcal{Z}^{c}(\underline{J})$ defined through

$$\mathcal{Z}^{c}(\underline{J}) = \Gamma(\underline{\varphi}) + \int (\underline{J}\cdot\underline{\varphi})(x)\,dx \Bigg|_{\left(\frac{\delta\Gamma}{\delta\underline{\varphi}} + \underline{J}\right)(x)=0} \tag{2}$$

coïncides with $\mathcal{L}^{cl}$ in the lowest approximation of a perturbative expansion in powers of $\hbar$, and, in higher orders, generates "proper" Feynman graphs.

Let us now consider a classical Lagrangian

$$\begin{aligned}\mathcal{L}^{cl}_{(1)}(\underline{J}) &\equiv \mathcal{L}_{(1)}(\underline{\varphi}) + \underline{J}\cdot\underline{\varphi} \\ &= \mathcal{L}_{inv}(\underline{\varphi}) - \frac{\mathcal{G}^2}{2\alpha} + \underline{J}\cdot\underline{\varphi}\end{aligned} \tag{3}$$

where $\mathcal{L}_{inv}(\underline{\varphi})$ is invariant under local abelian gauge transformations of the second kind ;

$$\delta\underline{\varphi}(x) = \int \frac{\delta\underline{\varphi}(x)}{\delta\Lambda(y)}\,\delta\Lambda(y)\,dy \tag{4}$$

$\mathcal{G}$ is a gauge function which breaks gauge invariance, and α is a numerical parameter, as they occur for instance in quantum electrodynamics. Noether's theorem yields the following Ward identity:

(5)

$$\int dx \left[\underline{J}(x) \cdot \frac{\delta \underline{\varphi}(x)}{\delta \Lambda(y)} - M(x,y) \frac{G(x)}{\alpha} \right] = 0$$

where the substitution

$$\underline{\varphi} = \frac{\delta Z^c(\underline{J})}{\delta \underline{J}} \tag{6}$$

has to be made, and where

$$M(x,y) = \frac{\delta G(x)}{\delta \Lambda(y)} \tag{7}$$

is the kernel of a field dependent differential operator of hyperbolic character whenever G is a perturbed version of the divergence of the gauge vector field associated with the gauge transformations under consideration. We shall from now on limit ourselves to this situation.

The Ward identity (5) can conveniently be solved for G upon introducing scalar charged Faddeev-Popov $(\phi\,\pi)$ ghost fields [13] and the corresponding sources into the initial Lagrangian

$$\begin{aligned} \mathcal{L}^{cl}(x) &= \mathcal{L}_{inv}(x) - \frac{1}{\alpha}\left(\frac{G^2(x)}{2} + \int dy\, \bar{c}(y) M(x,y) c(x) \right) + \left[\underline{J} \cdot \underline{\varphi} + \bar{\xi} c + \xi \bar{e} \right](x) \\ &= \mathcal{L}(\underline{\varphi}, c, \bar{c})(x) + \left[\underline{J} \cdot \underline{\varphi} + \bar{\xi} c + \xi \bar{e} \right](x) \end{aligned} \tag{8}$$

The Fermi statistics conventionally assigned to these fields while preserving locality introduces new sources of indefinite metric into the quantum interpretation of such a system and, at the same time exhibits crucial properties connected with the structure of the gauge transformations, which are best observed in the non abelian case described in Appendix I . The new Ward identity reads :

$$W(x)(Z^c) = \int dy \left[\underline{J}(y) \cdot \frac{\delta \underline{\varphi}(y)}{\delta \Lambda(x)} - \frac{1}{\alpha} M(y,x) G(y) - \frac{1}{\alpha} \int dz\, \bar{c}(z) \frac{\delta M(y,z)}{\delta \Lambda(x)} c(y) \right] = 0 \tag{9}$$

Integrating through $\bar{e}$ yields the so-called Slavnov identity [3], which, in the present, abelian, case, reads :

$$S(Z_c) = \int dx \left[\bar{\xi}(x) G(x) + \int dy\, \underline{J}(x) \cdot \frac{\delta \underline{\varphi}(x)}{\delta \Lambda(y)}\, \bar{c}(y) \right] = 0 \tag{10}$$

where use has been made of the equations of motion for the $\phi\,\pi$ fields,

and of their anticommutativity, whereby the last term in the Ward identity (9) drops out in view of the abelianness of the gauge transformations. In the non abelian case treated in Appendix I, this last term contributes however in a way which is characterized in terms of the structure constants of the Lie algebra involved.

The Slavnov identity can be interpreted as expressing the invariance of $\mathcal{L}$ under the following transformations of the supergauge type[14], which we shall call Slavnov transformations :

$$\begin{aligned} \delta_\lambda \underline{\varphi}(x) &= \lambda \int \frac{\delta \underline{\varphi}(x)}{\delta \Lambda(y)} \bar{c}(y)\, dy \\ \delta_\lambda c(x) &= \lambda\, \zeta(x) \\ \delta_\lambda \bar{c}(x) &= 0 \end{aligned} \tag{11}$$

where λ is an infinitesimal, space time independent, gauge parameter of the Fermi type. The vanishing of the variation of $\bar{c}$ is due to the abelian character of the gauge transformations and is suitably altered in the non abelian case as shown in Appendix I . The Fermi character of the $\phi\pi$ field linearizes the gauge "group" since

$$\begin{aligned} \delta_{\lambda_1} \delta_{\lambda_2} \underline{\varphi}(x) &= 0 \\ \delta_{\lambda_1} \delta_{\lambda_2} \bar{c}(x) &= 0 \\ \delta_{\lambda_1} \delta_{\lambda_2} c(x) &= \lambda_1 \lambda_2 \int M(y,x)\, \bar{c}(y)\, dy \end{aligned} \tag{12}$$

so that

$$S^2(Z_c) \equiv \int dx\, dy\, \bar{\xi}(y)\, M(y,x)\, \bar{c}(x) = 0 \tag{13}$$

and

$$\left(S \cdot S^2 - S^2 \cdot S\right)(Z_c) = 0 \tag{14}$$

One should realize the lack of equivalence, in general, between the Ward identity (5) and the Slavnov identity (10) : if one adds to $\mathcal{L}$ a breaking term of the form

$$-\frac{B}{\alpha} \tag{15}$$

such that

$$\frac{\delta B(x)}{\delta \Lambda(y)} = \mathcal{N}(x,y)\, G(x) \tag{16}$$

where $\mathcal{N}$ is a possibly field dependent differential operator which does not upset the hyperbolic character of $\mathcal{M}$, the Lagrangian

$$\mathcal{L}_B^{a.} = \mathcal{L}_{inv}(x) - \frac{1}{\alpha}\left(\frac{G^2(x)}{2} + B(x) + \int dy\, \bar{c}(y)\left(\mathcal{M}(x,y) + \mathcal{N}(x,y)\right)c(x)\right)$$
$$+ \underline{J}(x)\cdot\underline{\varphi}(x) + \bar{\xi}(x)\, c(x) + \xi(x)\, \bar{c}(x) \tag{17}$$

will lead to the same Slavnov identity whereas the Ward identity is modified according to

$$W^{(B)}(x)[Z_c] = \int dy \left[\underline{J}(y)\cdot\frac{\delta\underline{\varphi}(y)}{\delta\Lambda(x)} - \frac{1}{\alpha}\left(\mathcal{M}(y,x) + \mathcal{N}(y,x)\right)G(y)\right.$$
$$\left. - \frac{1}{\alpha}\int dz\, \bar{c}(z)\,\frac{\delta\left(\mathcal{M}(y,z) + \mathcal{N}(y,z)\right)}{\delta\Lambda(x)}\, c(y)\right] = 0 \tag{18}$$

This pathological situation is due to the abelianness of the gauge transformations which insures the absence from the Slavnov identity of a contribution involving the last term of the Ward identity.

A concrete example of this phenomenon will be given in the context of the abelian Higgs Kibble model treated within a family of linear, charge conjugation odd gauges.

The basic fields and sources are given in Table I

FIELD	BEHAVIOUR UNDER CHARGE CONJUGATION	SOURCE
φ_1	even	J_1
φ_2	odd	J_2
A_μ	odd	J_μ
c } $\phi\pi$ ghosts	even	ξ
$\bar{c}$ }	even	$\bar{\xi}$

TABLE I : Fields and Sources

One may choose for the Slavnov transformation:

$$\begin{aligned}
\delta\varphi_1 &= -\lambda e_1^\circ \varphi_2 \bar{c} \\
\delta\varphi_2 &= +\lambda e_2^\circ (\varphi_1 + v^\circ)\bar{c} \\
\delta A_\mu &= \lambda \partial_\mu \bar{c} \\
\delta c &= \lambda (a^\circ \partial_\mu A_\mu + \rho^\circ \varphi_2) \\
\delta \bar{c} &= 0
\end{aligned} \tag{19}$$

where v° is a field translation parameter e_1°, e_2° are charge parameters, a° and ρ° characterize the gauge function . The corresponding Slavnov identity reads:

$$S(Z_a) \equiv \int dx \left\{ J_\mu \partial_\mu \frac{\delta Z_c}{\delta \xi} - e_1^\circ J_1 \frac{\delta Z_c}{\delta J_2} \frac{\delta Z_c}{\delta \xi} + e_2^\circ J_2 \left(\frac{\delta Z_c}{\delta J_1} \frac{\delta Z_c}{\delta \xi} + v^\circ \frac{\delta Z_c}{\delta \xi} \right) - \bar{\xi} \left[a^\circ \partial_\mu \frac{\delta Z_c}{\delta J_\mu} + \rho^\circ \frac{\delta Z_c}{\delta J_2} \right] \right\}(x) = 0 \tag{20}$$

Eq(20) can be linearized by introducing into the Lagrangian the source terms:

$$\eta_1 (z_1 \varphi_1 \bar{c} + z_1' \bar{c}) + \eta_2 z_2 \varphi_2 \bar{c} \tag{21}$$

where η_1, η_2 are Fermi type sources:

$$S(Z_c) \equiv \mathcal{J} Z_c = 0 \tag{22}$$

Now Z_c also depends on η_1, η_2 whereas the Lagrangian is the partial Legendre transform of Z_c with respect to $J_1, J_2, J_\mu, \xi, \bar{\xi}$.

$\mathcal{J}$ is now a linear functional partial differential operator of the form :

$$\mathcal{J} = \int dx \Big[J_\mu \partial_\mu \delta_\xi - e_1 J_1 \delta_{\eta_2} + e_2 J_2 \delta_{\eta_1} + m J_2 \delta_{\bar{\xi}} - \bar{\xi} (a \partial_\mu \delta_{J_\mu} + \rho \delta_{J_2}) \Big](x) \tag{23}$$

The transformation law (19) is easily converted by translation and renormalization of the field variables into the more conventional one

$$\begin{aligned} \delta \varphi_1 &= -\lambda e \varphi_2 \bar{c} \\ \delta \varphi_2 &= \lambda e (\varphi_1 + v) \bar{c} \\ \delta A_\mu &= \lambda \partial_\mu \bar{c} \\ \delta c &= \lambda (\partial_\mu A_\mu + \rho \varphi_2) \end{aligned} \tag{24}$$

($e_1^\circ = e_2^\circ = e$, $v = v^\circ$, $\rho = \rho^\circ$, $a^\circ = 1$), where we keep however a field translation parameter explicit .

One may ask oneself what is the most general Lagrangian of the renormalizable type which is invariant under such a transformation , even under charge conjugation and carrying zero $\phi\pi$ charge .

This problem is a purely algebraic one .The most general Lagrangian of the renormalizable type which carries the vacuum quantum numbers is, up to a divergence , a linear combination of the following twenty six monomials:

0) φ_1
1) φ_1^2
2) φ_2^2
3) φ_1^3
4) $\varphi^2\varphi_1$
5) φ_1^4
6) φ_2^4
7) $\varphi_1^2\varphi_2^2$
8) $\partial_\mu\varphi_1\,\partial_\mu\varphi_1$
9) $\partial_\mu\varphi_2\,\partial_\mu\varphi_2$
10) $\varphi_2\,\partial_\mu A_\mu$
11) $A_\mu\varphi_1\partial_\mu\varphi_2$
12) $A_\mu\varphi_2\,\partial_\mu\varphi_1$
13) $A_\mu A_\mu$
14) $A_\mu A_\mu\varphi_1$
15) $A_\mu A_\mu\varphi_1^2$
16) $A_\mu A_\mu\varphi_2^2$
17) $(\partial_\mu A_\mu)^2$
18) $(\partial_\mu A_\nu-\partial_\nu A_\mu)(\partial_\mu A_\nu-\partial_\mu A_\nu)$
19) $(A_\mu A_\mu)^2$
20) $\bar c\,c$
21) $\bar c\,\varphi_1 c$
22) $\bar c\,\varphi_1^2 c$
23) $\bar c\,\varphi_2^2 c$
24) $\bar c\,A_\mu A_\mu c$
25) $\bar c\,\Box\,c$ (25)

Its variation $\delta\mathcal{L}$ under the Slavnov transformation (24) is of maximal dimension five, carrying the $\Phi\pi$ charge of a $\bar c$ ghost, odd under charge conjugation . It is therefore a combination of the 23 monomials:

1) $\bar c\varphi_2$
2) $\bar c\varphi_1\varphi_2$
3) $\bar c\varphi_2^3$
4) $\bar c\varphi_1^2\varphi_2$
5) $\bar c\,\Box\varphi_2$
6) $\bar c\varphi_1^3\varphi_2$
7) $\bar c\varphi_1\varphi_2^3$
8) $\bar c\varphi_2\Box\varphi_1$
9) $\bar c\varphi_1\Box\varphi_2$
10) $\bar c\,\partial_\mu\varphi_1\partial_\mu\varphi_2$
11) $\bar c\,\partial_\mu A_\mu$
12) $\bar c\,A_\mu\partial_\mu\varphi_1$
13) $\bar c\,\partial_\mu A_\mu\varphi_1$
14) $\bar c\,\partial_\mu A_\mu\varphi_1^2$
15) $\bar c\,A_\mu\varphi_1\partial_\mu\varphi_1$
16) $\bar c\,\partial_\mu A_\mu\varphi_2^2$
17) $\bar c\,A_\mu\varphi_2\partial_\mu\varphi_2$
18) $\bar c\,\Box\,\partial_\mu A_\mu$
19) $\bar c\,A_\mu\partial_\mu\bar c\,c$
20) $\bar c\,A_\mu A_\nu\partial_\mu A_\nu$
21) $\bar c\,A_\mu A_\mu\varphi_2$
22) $\bar c\,A_\mu A_\mu\varphi_1\varphi_2$
23) $\bar c\,A_\mu A_\mu\partial_\mu A_\mu$ (26)

One can however verify that the last three monomials can never occur as variations of some monomials in Eq.(25) whereas the first twenty are such variations. It follows that the requirement that $\mathcal{L}$ be invariant under Slavnov transformations is expressed via a homogeneous linear system of twenty equations whose unknowns are the coefficients of the twenty six monomials listed in Eq.(25) As a result, the most general invariant $\mathcal{L}$ can be written as a linear combination of the following six terms:

$$
\begin{aligned}
&1)\quad G_{\mu\nu}\,G_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu)\\
&2)\quad (D_\mu \varphi)^* D_\mu \varphi\\
&3)\quad \varphi^* \varphi\\
&4)\quad (\varphi^* \varphi)^2\\
&5)\quad \frac{g^2}{2} + \bar{c}\,\frac{\delta g}{\delta \Lambda}\,c\\
&6)\quad \frac{A_\mu A_\mu}{2} - \bar{c}\,c + \frac{\rho}{e}\,\varphi_1
\end{aligned}
\tag{27}
$$

where

$$
\varphi = \frac{\varphi_1 + v + i\varphi_2}{\sqrt{2}} \qquad g = \partial_\mu A_\mu + \rho\varphi_2
$$

$$
D_\mu = \partial_\mu - ieA_\mu \qquad \frac{\delta g}{\delta \Lambda} = \Box + \rho e v + \rho e \varphi_1
\tag{28}
$$

In other words, is of the form

$$
\begin{aligned}
\mathcal{L} = &-\frac{Z_A}{4} G_{\mu\nu}\,G_{\mu\nu} + Z_1 (D_\mu \varphi)^* D_\mu \varphi + \mu^2 \varphi^* \varphi\\
&- g(\varphi^* \varphi)^2 - \frac{1}{\alpha}\left(\frac{g^2}{2} + \bar{c}\,\frac{\delta g}{\delta \Lambda}\,c\right)\\
&+ \beta\left(\frac{1}{2} A_\mu A_\mu - \bar{c}\,c + \frac{\rho}{e}\,\varphi_1\right)
\end{aligned}
\tag{29}
$$

The last term which is conspicuously absent from the classical Higgs Kibble Lagrangian has precisely to do with the phenomenon previously alluded to. Its presence violates spontaneous breakdown without spoiling the Slavnov identity . As we shall see later , its absence can be imposed by requiring suitable normalization conditions on the Green functions which allow to convert the unphysical parameters $Z_A, Z_1, \mu^2, g, \alpha, \rho, \beta$ into parameters that are needed to interpret the theory in terms of particles .

In terms of the variables appropriate to the case of broken symmetry, $\mathcal{L}$(29) can rewritten as:

$$\mathcal{L} = -\frac{Z_A}{4} G_{\mu\nu} G_{\mu\nu} + Z_1 \left[\partial_\mu \varphi_1 \partial_\mu \varphi_1 + \partial_\mu \varphi_2 \partial_\mu \varphi_2 \right.$$

$$+ 2e A_\mu \left(\varphi_2 \partial_\mu \varphi_1 - (\varphi_1 + v) \partial_\mu \varphi_2 \right) + e^2 A_\mu A_\mu \left((\varphi_1 + v)^2 \right. \qquad (30)$$

$$\left. \left. + \varphi_2^2 \right) \right] + \mu^2 \left[(\varphi_1 + v)^2 + \varphi_2^2 \right] - g \left[(\varphi_1 + v)^2 + \varphi_2^2 \right]^2$$

$$- Z_4 \left[\frac{(\partial_\mu A_\mu + g \varphi_2)^2}{2} + \bar{c} \left(\Box + g e (\varphi_1 + v) \right) c \right]$$

$$+ \beta \left[\frac{A_\mu A_\mu}{2} - \bar{c} c + \frac{g}{e} \varphi_1 \right]$$

We shall now impose the following normalization conditions, which for reasons to be explained , we split into two groups:

Unphysical :	$\langle \varphi_1 \rangle = 0$	(0)	
	$\Gamma_{c\bar{c}} (p^2 = m_4^2) = 0$	(I)	
	$\Gamma'_{c\bar{c}} (p^2 = m_4^2) = \frac{1}{\alpha}$	(2)	
Physical	$\Gamma_{\varphi_1 \varphi_1} (p^2 = M^2) = 0$	(3)	(3I)
	$\Gamma'_{\varphi_1 \varphi_2} (p^2 = M^2) = z_{\varphi_1} = 1$	(4)	
	$\Gamma_{A^T A^T} (p^2 = m^2) = 0$	(5)	
	$\Gamma'_{A^T A^T} (p^2 = m^2) = z_A = 1$	(6)	
	$\Gamma_{A^T A^T \varphi_1} (m^2, m^2, M^2) = 2 \varepsilon m$	(7)	
	$\Gamma^{-1}_{c\bar{c}} \det \begin{pmatrix} \Gamma_{A^L A^L} & \Gamma_{A^L \varphi_2} \\ \Gamma_{A^L \varphi_2} & \Gamma_{\varphi_2 \varphi_2} \end{pmatrix} \Big|_{p^2 = m_4^2} = 0$	(8)	

Here A^T (resp A^L) denotes the transverse (resp. longitudinal part) of A; expressed in terms of the parameters specifying $\mathcal{L}$, these conditions read:

$$\langle \varphi_1 \rangle = 0 = 2 v \mu^2 - 4 v^3 g + \frac{g \beta}{e} \qquad (0)$$

$$m_4^2 = g e v + \beta \alpha \qquad (I)$$

$$Z_G = \frac{1}{\alpha} \tag{2}$$

$$M^2 = \frac{2}{Z_1}(6g v^2 - \mu^2) \tag{3}$$

$$Z_2 = Z_{\varphi_1} = 1 \tag{4}$$

$$m^2 = [Z_1 e^2 v^2 + \beta] Z_A^{-1} \tag{5}$$

$$Z_A = Z_A = 1 \tag{6}$$

$$e = \varepsilon \tag{7}$$

(32)

$$\Gamma^{-1}_{c\bar{c}} \det \begin{pmatrix} \Gamma_{A^L A^L} & \Gamma_{A^L \varphi_2} \\ \Gamma_{A^L \varphi_2} & \Gamma_{\varphi_2 \varphi_2} \end{pmatrix} \Bigg|_{p^2 = m_G^2}$$

$$= -\frac{1}{\alpha} \frac{1}{p^2 - \rho e v - \beta\alpha} \left((p^2 - \rho e v - \beta\alpha)\left(p^2 - \rho e v - \frac{\beta\rho}{ev}\right) \right) \Bigg|_{p^2 = m_G^2}$$

$$= \beta\left(\frac{\rho}{ev} - \alpha\right) = 0 \tag{8}$$

This last normalization condition is well defined because the $\phi\pi$ ghost mass turns out to be degenerate with at least one of the coupled (A^L, φ_2) ghost system . This is a consequence of the Slavnov identity , as shown in appendix III . On the other hand complete degeneracy of the ghost masses is precisely the condition for spontaneous breakdown , ($\beta = 0$), except if $\rho = \alpha e v$, which characterizes the restricted t' Hooft gauge, excluded here and eventually recovered by a limiting procedure

The system is an algebraic system which is invertible and allows to solve for the coefficients in the Lagrangian in terms of the parameters occuring in the normalization conditions. This leads to a particle interpretation of the theory in a Fock space carrying an indefinite metric due to the Fermi character of the $\phi\pi$ ghosts and the non positive definiteness of the(A^L, φ_2) coupled propagator matrix .

One can easily generalize this analysis to the case where $e_1^0 \neq e_2^0$, $a^0 \neq 1$ where the theory is again determined by the Slavnov identity and normalization conditions, e_2^0 and a^0 being left free . Although the corresponding algebra is not illuminating and will not be reported here , the

possibility of such a generalization should be kept in mind for further reference .

We are now able to describe the scattering theory : the Fock space is determined by the quadratic part of $\mathcal{L}$, the corresponding in fields being solutions of the derived Euler Lagrange equations . Within this Fock space we may select a physical subspace generated from vacuum by application of ($\varphi^{\dagger}_{1,in}$, $A^{T}_{\mu,in}$). Physical states should actually be equivalence classes of such states modulo some zero norm states whose structure will be mentioned later in connection with the questions of the unitarity of the physical S operator and of the existence of physical local observables

The restriction to the above defined physical subspace of the connected scattering operator is given by the LSZ formula :

$$S^{c}_{phys} = :\exp\Big[i\int dx\,dy\Big(\varphi^{\dagger}_{1,in}(x)\,K_{1}(x,y)\frac{\delta}{\delta J_{1}(y)} + A^{T}_{\mu,in}(x)\,K_{\mu\nu}(x,y)\frac{\delta}{\delta J_{\nu}(y)}\Big)\Big]: Z^{c}[\underline{J}]\Big|_{J_{\mu}=J_{1}=J_{2}=\bar{\xi}=\xi=0}$$
$$\equiv \sum_{phys} Z^{c}(\underline{J})\Big|_{\underline{J}=0} \qquad (33)$$

where , in view of Eq(3I,3,4,5,6)

$$K_{1}(x,y) = (\Box + M^{2})\,\delta(x-y)$$
$$K_{\mu\nu}(x,y) = (\Box\, g_{\mu\nu} - \partial_{\mu}\partial_{\nu} + m^{2} g_{\mu\nu})\,\delta(x-y) \qquad (34)$$

It is typical of the spontaneously broken theory that the physical scattering operator does not depend on the parameters which specify the gauge . In other words,

$$\frac{\partial S_{phys}}{\partial \rho} = 0 \quad , \quad \frac{\partial S_{phys}}{\partial \alpha} = 0 \qquad (35)$$

The first relation can be proved as follows:

$$\frac{\partial S^{c}_{phys}}{\partial \rho} = \sum_{phys} \frac{\partial Z^{c}[J]}{\partial \rho}\Big|_{J=0}$$
$$= \sum_{phys} \Big(-\frac{1}{\alpha}\Big)\int dx\,\big[\,\rho\,\varphi_{2} + \bar{c}\,e^{0}_{2}(\varphi_{1}+v^{0})c\,\big](x)\Big|_{\underline{J}=0} = 0 \qquad (36)$$

since the expectation value of $\mathcal{G}$ between physical states vanishes because of the Slavnov identity and those of c and $\bar{c}$ because of $\phi\pi$ charge conservation Similarly,

$$\frac{\partial S^c_{phys}}{\partial \alpha} = \sum_{phys} \frac{\partial Z^c[J]}{\partial \alpha}\Big|_{J=0} \tag{37}$$

$$= \sum_{phys} \frac{1}{\alpha^2}\int dx \left(\frac{\mathcal{G}^2}{2} + \bar{c}\, m\, c\right)(x)\Big|_{J=0}$$

$$= \sum_{phys} \frac{1}{\alpha^2}\int dx \left(\frac{\mathcal{G}^2 + \bar{c}\, m\, c}{2} + \bar{\xi}\, c\right)(x)\Big|_{J=0} = 0$$

This concludes our review of the tree approximation .

II PERTURBATION THEORY TO ALL ORDERS: THE SLAVNOV IDENTITIES

The extension of the model beyond the tree approximation , proceeds in the spirit of the BPHZ [5] renormalization scheme, via an effective Lagrangian of the form

$$\mathcal{L}^{eff}(\varphi, J, \eta) = \mathcal{L}_4^{eff}(\varphi) + \eta_1 (z_1 N_2[\bar{c}\varphi_1] + z_1' \bar{c})$$

$$+ \eta_2 z_2 N_2[\bar{c}\varphi_2] + J_1\varphi_1 + J_2\varphi_2 + J_\mu A_\mu$$

$$+ \bar{\xi} c + \xi \bar{c}$$

$$= \mathcal{L}^{eff}(\varphi, \eta) + J\varphi \tag{38}$$

The corresponding Green functionals (39)

$$Z(J, \eta) = \langle T \exp\left[\frac{i}{\hbar}\int \mathcal{L}_{int}^{eff}(\varphi, J, \eta)(x)\, d^4x\right]\rangle$$

and (40)

$$Z^c(J, \eta) = \frac{\hbar}{i} \ln Z(J, \eta) = \frac{\hbar}{i}\langle T \exp\left[\frac{i}{\hbar}\int d^4x\, \mathcal{L}_{int}^{eff}(\varphi, J, \eta)(x)\right]\rangle^c$$

are expressed in terms of Feynman graphs in which the propagators are defined by the quadratic part $\mathcal{L}^0$ of $\mathcal{L}$ (Eqs.(30,31)), and the vertices are given by

$$\mathcal{L}_{int}^{eff} = \mathcal{L}^{eff}(\underline{\varphi}, \underline{J}, \underline{\eta}) - N_4 \mathcal{L}_0 \tag{41}$$

The substraction procedure which defines the time ordering symbol T in Eqs.(39,40) being specified by the N prescriptions indicated in Eq.(38). The coefficients of the Wick monomials in $\mathcal{L}^{eff}$ are to be considered as formal power series in $\hbar$, and , of course, $\mathcal{L}^{eff}$ should coïncide in zeroth order with (Eqs.(31))

We shall also clearly restrict ourselves to effective Lagrangians even under charge conjugation and carrying no $\phi\pi$ charge .

One can furthermore immediately specialize Eq.(38) by making the choice

$$z_1 = z_2 = 1 \qquad z_1' = 0 \tag{42}$$

which corresponds to fixing normalization conditions on the fields coupled to $\underline{\eta}$. We can also define $\mathcal{L}^{eff}(\underline{\varphi})$ so that no linear term is present , thus automatically fulfilling the normalization condition (31,0)

$$\langle \varphi_1 \rangle = 0 \tag{31,0}$$

We shall have however to keep in mind in the following that the allowed class of Lagrangians is that written down in Eq.(38) and $\mathcal{L}^{eff}(\underline{\varphi})$ is a linear combination of 25 terms which are listed in Eq.(25) (excluding Eq.(25,0) in view of Eq.(31,0)

The question is now whether one can determine $\mathcal{L}^{eff}$ so that $Z^c(\underline{J}, \underline{\eta})$ fulfills a renormalized Slavnov identity :

$$\mathcal{S} Z^c(\underline{J}, \underline{\eta}) = \int dx \Big[J_\mu \partial_\mu \delta_\xi - \bar{e}_1 J_1 \delta_{\eta_2} + \bar{e}_2 J_2 \delta_{\eta_1} + \bar{m} J_2 \delta_\xi$$

$$- \bar{\xi} \left(\bar{a} \partial_\mu \delta_{J_\mu} + \bar{\rho} \delta_{J_2} \right) \Big](x)\, Z^c(\underline{J}, \underline{\eta}) = 0 \tag{43}$$

where $\bar{a}$, $\bar{\rho}$, $\bar{e}_1$, $\bar{e}_2$, $\bar{m} = \bar{e}_2 \bar{\tau}$ are formal power series in $\hbar$.

We shall eventually require that the normalization conditions (31.) be fulfilled .

Now, according to Lam's [7] renormalized action principle , the Slavnov identity (43) expresses the invariance of the effective Lagrangian under the renormalized Slavnov transformation

$$\begin{aligned}
\delta\varphi_1 &= \lambda \bar{e}_2 N_2[\varphi_2 \bar{e}] \\
\delta\varphi_2 &= -\lambda\left(\bar{e}_1 N_2[\varphi_1 \bar{e}] - \bar{m}\,\bar{e}\right) \\
\delta A_\mu &= \lambda \partial_\mu \bar{e} \\
\delta c &= \lambda\left(\bar{a}\,\partial_\mu A_\mu + \bar{f}\,\varphi_2\right) \\
\delta \bar{e} &= 0
\end{aligned} \tag{44}$$

Indeed performing on an arbitrary effective Lagrangian the quantum variation (44) according to the quantum action principle yields:

$$\delta Z^e(J,\eta) = \Delta Z^c(J,\eta) \tag{45}$$

where the left hand side comes from the variation of the source terms, and where $-\Delta$ is precisely the insertion of the quantum variation of the effective Lagrangian $\mathcal{L}^{eff}(\varphi,\eta)$. It is a consequence of Lam's analysis that:

$$\Delta = -\int dx\, N_5\left[s\mathcal{L}^{eff}(\varphi,\eta) + \hbar Q\right](x) \tag{46}$$

where $s\mathcal{L}^{eff}$ is the naïve variation of the Lagrangian , and $\hbar Q$ sums up the quantum corrections. Because of power counting and selection rules. Δ is a linear combination of twenty three monomials listed in Eq.(26), the coefficients being formal power series in $\hbar$ and in the coefficients of $\mathcal{L}^{eff}$ as well as in those appearing in Eq.(44) . The symmetry condition we are looking for is

$$\Delta = 0 \tag{47}$$

It can be partially satisfied by requiring that the coefficients of the 20 first monomials vanish to all orders in $\hbar$ the parameters of the Slavnov identity being left arbitrary to all orders . The argument is that if

$$\mathcal{L}^{eff} = \underline{c}_4 \underline{\mathcal{L}}^4 + \underline{c}_b \underline{\mathcal{L}}^b \tag{48}$$

where: $$s\underline{\mathcal{L}}^4 = 0 \ , \quad s\underline{\mathcal{L}}^b \neq 0 \tag{49}$$

we can write:

$$\underline{\Delta} = \underline{d}_b s\underline{\mathcal{L}}^b + \hbar \mathcal{R} \tag{50}$$

where $\underline{d}_b = \underline{c}_b + \hbar \underline{\Phi}_b$, $\underline{\Phi}_b$ being a formal power series in $\hbar, \underline{c}_4, \underline{c}_b$ and the coefficients $\underline{s}$ of the Slavnov identity , and the quantum correction $\hbar\mathcal{R}$ is not of the form $s\underline{\mathcal{L}}^b$, namely it involves the last three monomials in Eq(26). By the implicit function theorem for formal power series , (cf. Appendix II) the system:

$$\underline{c}_b + \hbar \underline{\Phi}_b(\hbar, \underline{c}_4, \underline{c}_b, \underline{s}) = 0 \tag{51}$$

is soluble for $\underline{c}_b$.

$\mathcal{L}^{eff}$ is thus now determined in terms of five parameters (because of (31,0)) and of the five coefficients involved in the Slavnov identity which now reads

$$sZ^c(\underline{J}, \underline{\eta}) = (c_1\Delta_1 + c_2\Delta_2 + c_3\Delta_3) Z^c(\underline{J}, \underline{\eta}) \tag{52}$$

where $\Delta_1, \Delta_2, \Delta_3$ (previously numbered 21,22,23) are the last three terms in (26), affected with the N_5 prescription .

Now, obviously, the right hand side (52) has to fulfill the compatibility condition implied by the structure of the left hand side(cfEqs.(12,13) namely:

$$\begin{aligned} s^2 Z^c &\equiv -\int dx \left[\sum_\xi \left((\bar{a}\Box + \bar{\rho}\bar{m}) \delta_{\xi\xi} + \bar{\rho}\bar{e}_2 \delta_{\eta_1} \right) \right](x) \, Z^c \\ &\equiv -\int dx \left[\sum_\xi (\bar{m} s_{\xi\xi}) \right](x) \, Z^c \\ &= s(c_1\Delta_1 + c_2\Delta_2 + c_3\Delta_3) Z^c \\ &= \left[s, c_1\Delta_1 + c_2\Delta_2 + c_3\Delta_3 \right] Z^c \end{aligned} \tag{53}$$

Now : $[\mathcal{S}_e, \Delta_i] Z^c = [s\Delta_i + \hbar P_i] Z^c$ (54)

where $s\Delta_i$ is the naïve variation of the monomial Δ_i under a Slavnov transformation, to which dimension six is assigned , whereas P_i is a dimension six insertion , carrying two $\bar{c}$ charges, even under charge conjugation and whose coefficients are formal power series in $\hbar$ and in the so far undetermined power series coefficients occuring in $\mathcal{L}^{eff}$ and $\mathcal{S}$, as a consequence of Zimmermann's reduction formulae.

On the other hand , the $\Phi\pi$ ghost equation of motion is of the form :

$$\left(\bar{n}\mathcal{S}_{\bar{\xi}}\right)(x) Z^c = \left[(\bar{\alpha}\Box + \bar{\mu}^2)\delta_{\bar{\xi}} + d\,\delta_{\eta_1} + \vec{f}\cdot\delta_{\vec{\beta}}\right](x) Z^c\Big|_{\vec{\beta}=0} = \bar{\alpha}\bar{\xi}(x) \quad (55)$$

where $\bar{\alpha}, \bar{\mu}, d, \vec{f}, \bar{\alpha}$ are formal power series, and $\vec{\beta}$ is a source coupled to

$$\vec{\bar{\mathcal{C}}} = \left\{ N_3[\varphi_1^2 \bar{c}],\ N_3[\varphi_2^2 \bar{c}],\ N_3[A_\nu A_\nu \bar{c}] \right\}$$

Thus, integrating Eq.(52) through $\bar{\xi}$, one gets

$$\int dx \left[\bar{\xi}(\bar{n}\mathcal{S}_{\bar{\xi}})\right](x) Z^c = 0 \quad (56)$$

Noticing that

$$\int dx \left(\bar{\xi}\mathcal{S}_{\bar{\xi}}\right)(x) Z^c = 0 \quad (57)$$

as a consequence of the invariance of $\mathcal{L}^{eff}(\underline{J}, \underline{\eta})$ under the variation

$$\delta c = \partial \bar{c}$$

and substracting Eq.(56) from Eq.(53) yields :

$$\int_{\mathcal{S}} dx \left[\bar{\xi}\left((\bar{\rho}\bar{e}_2 - d)\delta_{\eta_1} - \vec{f}\cdot\delta_{\vec{\beta}}\right)\right](x) Z^c(\underline{J}, \underline{\eta}, \vec{\beta})\Big|_{\vec{\beta}=0}$$
$$= \sum_{1}^{3} {}_i\, e_i \left(s\Delta_i + \hbar P_i\right) Z^c(\underline{J}, \underline{\eta}) \quad (58)$$

We now express Z^c in terms of Γ by Legendre transform , thus obtaining :

$$\int dx \left[(d - \bar{\rho}\bar{e}_2)\delta_c \Gamma \delta_{\eta_1} \Gamma + \vec{f}\,\delta_e \Gamma \delta_{\vec{\beta}} \Gamma\right](x)\Big|_{\vec{\beta}=0}$$
$$= \sum_{1}^{3} {}_i\, e_i \left(s\Delta_i + \hbar P_i\right)\Gamma = 0\Big|_{\vec{\beta}=0} \quad (59)$$

Let us now write

$$\Gamma = \mathcal{L}^{eff}(\underline{J},\underline{\eta}) + \vec{\beta}\cdot\vec{\vec{e}} + \hbar\,\Gamma' \qquad (60)$$

where it has been explicitly noted that the corrections to $\mathcal{L}^{eff}$ occuring in Γ are necessarily radiative corrections. Eq.(59) can be cast into the form:

$$\int dx \Big\{ \Box\bar{c}\Big[(d-\bar{\rho}\bar{e}_2)\varphi_1\bar{c} + f_1\varphi_1^2\bar{c} + f_2\varphi_2^2\bar{c} + f_3 A_\mu A_\mu \bar{c} + 2c_1\bar{c}A_\mu\partial_\mu\bar{c}\varphi_2 + 2c_2 A_\mu\partial_\mu\bar{c}\varphi_1\varphi_2 + c_3\big(2\bar{c}\partial_\mu\bar{c}A_\mu\partial_\nu A_\nu + \bar{c}A_\mu A_\mu \Box\bar{c}\big)\Big]\Big\}(x)$$
$$= \hbar\,\Phi(d-\bar{\rho}\bar{e}_2, f_1, f_2, f_3, c_1, c_2, c_3) \qquad (61)$$

where Φ is a functional of the fields which is linear in the indicated arguments and lumps together contributions from Γ' and from the $P_i's$. Differenciating in turn Eq.(61) with respects to the fields occuring in each indicated monomial, and setting all fields equal to zero, yields, in view of the independance of these monomials:

$$\begin{aligned}
d-\bar{\rho}\bar{e}_2 &= \hbar\,\phi_0(d-\bar{\rho}\bar{e}_2, \underline{f}, \underline{c}) \\
f_1 &= \hbar\,\phi_1(d-\bar{\rho}\bar{e}_2, \underline{f}, \underline{c}) \\
f_2 &= \hbar\,\phi_2(d-\bar{\rho}\bar{e}_2, \underline{f}, \underline{c}) \\
f_3 &= \hbar\,\phi_3(d-\bar{\rho}\bar{e}_2, \underline{f}, \underline{c}) \\
c_1 &= \hbar\,\psi_1(d-\bar{\rho}\bar{e}_2, \underline{f}, \underline{c}) \\
c_2 &= \hbar\,\psi_2(d-\bar{\rho}\bar{e}_2, \underline{f}, \underline{c}) \\
c_3 &= \hbar\,\psi_3(d-\bar{\rho}\bar{e}_2, \underline{f}, \underline{c})
\end{aligned} \qquad (62)$$

where Φ_i $(i=1,2,3)$, Ψ_i $(i=1,2,3)$ are linear in the indicated arguments, formal power series in $\hbar$ and in the remaining parameters. The situation occuring in the tree approximation and application of the theorem in appendix II yield:

$$\Delta - \bar{\xi}\bar{e}_2 = e_1 = e_2 = e_3 = f_1 = f_2 = f_3 = 0 \qquad (63)$$

Hence, the Slavnov identity holds, and , up to the mass term the $\Phi\pi$ equation of motion involves the same coefficients and monomials as those occuring in Δ^2. The equality of the two mass terms will be proved in appendix III in connection with the normalization conditions we shall now consider .

Namely, we shall show that the normalization conditions (31) can be fulfilled, whereby all parameters are determined except $\bar{a}$ and $\bar{e}_2$ Eq.(31,0) is already fulfilled. Next we try to impose Eq.(31,1,2,3,4,5,6,7). Looking at the algebraic system which is soluble in the tree approximation , we can apply once more the theorem of appendix II, because this system is perturbed as allowed by this theorem by higher order terms occuring in $\Gamma_{\varphi_1\varphi_1}$, $\Gamma_{A^T A^T}$, $\Gamma_{c\bar{c}}$, $\Gamma_{A^T A^T \varphi_1}$.

The last normalization condition (31,8) is more delicate : one has first to show, to all orders in $\hbar$ that $\Gamma_{c\bar{c}}^{-1}\left(\Gamma_{A^L A^L}\Gamma_{\varphi_2\varphi_2} - \Gamma_{A^L\varphi_2}^2\right)$ is finite at $p^2 = m_4^2$. The proof, based on the Slavnov identity and Eq.(31,2) is given in appendix III. As a by product, as previously announced, one obtains the last equation connecting Δ^2 and the $\Phi\pi$ equation of motion (Eq.(52)) namely the $\Phi\pi$ equation reads:

$$\left(\bar{\mathcal{M}}\,\delta_{\xi}\right)(x)\,Z^e = \left(\bar{m}\,\delta_{\xi}\right)(x)\,Z^e = \alpha\,\bar{\xi}(x) \qquad (64)$$

In conclusion , once the Slavnov identities and the normalization conditions have been fulfilled , there remain two free parameters, $\bar{a}$ and $\bar{e}_2$, which will not be specialized any further .

III- PERTURBATION THEORY TO ALL ORDERS: GAUGE INVARIANCE OF THE PHYSICAL SCATTERING OPERATOR .

The normalization conditions Eq.(31) allow to interpret the theory , in the sense of formal power series, within the Fock space defined in the tree approximation , and the formula giving S^c_{phys} in terms of $Z^c(J)$ (Eq.31), or similarly S_{phys} in terms of $Z(J)$ remains unchanged . For a technical reason which will appear later we shall from now on work with the non connected Green functional .

We now wish to evaluate

$$\frac{\partial S_{phys}}{\partial g} \quad , \quad \frac{\partial S_{phys}}{\partial \alpha}$$

Using Lowenstein's [6] renormalized action principle we see that

$$\frac{\hbar}{i}\frac{\partial}{\partial \lambda} Z(\underline{J},\underline{\eta}) = \Delta_\lambda Z(\underline{J},\underline{\eta}) \tag{65}$$

where λ is one of the parameters α, m_4^2 and Δ_λ is a dimension four insertion obtained by differenciating $\mathcal{L}^{eff}(\underline{\varphi},\underline{\eta})$ with respect to λ , namely an operation which alters infinitesimally $\mathcal{L}^{eff}(\underline{\varphi},\underline{\eta})$ within the class (38)

Using the Slavnov identity , we are going to show that Δ_λ can be written as

$$\Delta_\lambda = \sum_1^8{}_i\, c_\lambda^{0,i}\, \Delta_i^0 + \sum_1^6{}_i\, c_\lambda^{s,i}\, \Delta_i^s \tag{66}$$

where the Δ_i^0's $(i=1,..,8)$ are eight insertions such that

$$\sum_{phys} \Delta_i^0 Z(\underline{J},\underline{\eta})\Big|_{\underline{J}=\underline{\eta}=0} = 0 \tag{67}$$

and leaving unchanged the physical normalization condition (31,8)

The other physical normalization conditions (31. 3,4,5,6,7) are left unaltered as a consequence of Eq.(67). In the following, we shall call these insertions non physical. The Δ_i^s , $(i=1,\ldots,6)$, are six symmetric insertions, namely such that

$$\mathcal{S}\,\Delta_i^s Z(\underline{J},\underline{\eta}) = 0 \quad (i=1,\ldots,6) \tag{68}$$

Thus applying Eq.(65) to the physical normalization conditions (31,3,4,5,6,7,8) yields a linear homogeneous system of equations of the form

$$\sum_{1}^{6}{}_i \; C_\lambda^{S,i} \; \Delta_i^{S,j} = 0 \qquad (j = 3,4,5,6,7,8) \tag{69}$$

The forthcoming analysis shows that

$$\det \| \Delta_i^{S,j} \| \neq 0 \tag{70}$$

since this happens to be true in the tree approximation . Hence it follows that

$$C_\lambda^{S,i} = 0 \tag{71}$$

and the gauge invariance of the physical S-operator follows from Eq(67)

We now construct the decomposition of Δ_λ given in Eq.(67)

From the definition of Δ_λ we have

$$\partial_\lambda(\delta Z) = \partial_\lambda \delta Z + \frac{i}{\hbar} \delta \Delta_\lambda Z = 0 \tag{72}$$

so that

$$\frac{i}{\hbar}[\Delta_\lambda, S] = \partial_\lambda S \tag{73}$$

Thus we can write

$$\Delta_\lambda = \hat{\Delta}_\lambda + \Delta_\lambda^S \tag{74}$$

where $\hat{\Delta}_\lambda$ is a particular solution of Eq.(73) and Δ_λ^S is a symmetrical insertion

We shall first construct a non physical $\hat{\Delta}_\lambda$, and we shall show that any Δ_λ^S is a linear combination of nine symmetrical insertions three of which are non physical the remaining six satisfying Eq.(68)

Let us denote

$$\delta = \sum_{0}^{5}{}_i \; c_i \, \delta_i \tag{75}$$

where

$$\begin{aligned}
c_0 &= 1 & \delta_0 &= \int dx \left(J_\mu \partial_\mu \delta_\xi\right)(x) \\
c_1 &= -\bar{e}_1 & \delta_1 &= \int dx \left(J_1 \delta_{\eta_2}\right)(x) \\
c_2 &= \bar{e}_2 & \delta_2 &= \int dx \left(J_2 \delta_{\eta_1}\right)(x) \\
c_3 &= \bar{m} & \delta_3 &= \int dx \left(J_2 \delta_\xi\right)(x) \\
c_4 &= -\bar{a} & \delta_4 &= \int dx \left(\bar{\xi} \partial_\mu \delta_{J_\mu}\right)(x) \\
c_5 &= -\bar{\beta} & \delta_5 &= \int dx \left(\bar{\xi} \delta_{J_2}\right)(x)
\end{aligned} \tag{76}$$

So that Eq.(73) now reads:

$$\frac{i}{\hbar}\left[\hat{\Delta}_\lambda, S\right] = \sum_i^5 \partial_\lambda c_i \, s_i \qquad (77)$$

Now, there exists a basis of covariant non physical insertions Δ_i: i ($i=1,\ldots,5$) satisfying :

$$\frac{i}{\hbar}\left[\Delta_i, S\right] = s_i$$

Indeed let us consider :

$$Q_{i,\varepsilon} = \frac{\hbar^2 c_i}{\bar{\alpha}} \int dx \left(\delta_{g_i(\varepsilon)} \, \delta_{\xi(-\varepsilon)}\right)(x) \qquad (i=4,5) \qquad (78)$$

where the $c_i s_i$ are defined in Eq.(76) and $\bar{\alpha}$ in Eq.(55)

The symbols δ_{g_i} are defined by:

$$\delta_{g_4} = \partial_\mu \delta_{J_\mu}$$
$$\delta_{g_5} = \delta_{J_2} \qquad (79)$$

The indices $(\pm\varepsilon)$ indicate translations by the e.g. space like small vectors $\pm\varepsilon$.

We introduce the insertions:

$$\Delta_{i,\varepsilon} Z = S\, Q_{i,\varepsilon} Z = \left[S, Q_{i,\varepsilon}\right] Z \qquad (80)$$

and we have :

$$\frac{i}{\hbar}\left[\Delta_{i,\varepsilon}, S\right] Z = -\frac{i}{\hbar}\left[S^2, Q_{i,\varepsilon}\right] Z \qquad (81)$$

$$= -\frac{i}{\hbar}\left[Q_{i,\varepsilon}, \int dx \left[\bar{\xi}\left(\bar{m}\delta_\xi\right)\right](x) = \frac{\hbar}{i}\frac{c_i}{\bar{\alpha}}\int dx \left[\delta_{g_i(\varepsilon)}\left(\bar{m}\delta_\xi\right)(-\varepsilon)\right](x) = c_i \int dx \left[\bar{\xi}(-\varepsilon)\,\delta_{g_i(\varepsilon)}\right](x)$$

where the connection between S^2 and the $\phi\pi$ equation of motion (Eq.64) has been used .

It is shown in Appendix IV that, in the limit $\varepsilon \to 0$ the finite part Δ_i of $\Delta_{i,\varepsilon}$ has the same covariance as $\Delta_{i,\varepsilon}$, namely

$$c_i s_i Z(J,\eta) = \lim_{\varepsilon\to 0} c_i \int dx \left[\bar{\xi}(-\varepsilon)\,\delta_{g_i(\varepsilon)}\right](x)\, Z(J,\eta) \qquad (82)$$

It is furthermore shown, in appendix IV , that by substracting a symmetric insertion , which therefore does not alter the covariance $c_i s_i$, one obtains non physical insertions which we denote Δ_i^0 .

We now look for other non physical insertions which are easily obtained by applying the renormalized action principle (6)

The following variations whose covariances are indicated provide us with the desired insertions :

I) $\delta c \propto c$ yields the insertion

$$\Delta_c = \frac{\hbar}{i} \int dx \, (\Xi \delta_\Xi)(x) \tag{83}$$

with

$$\frac{i}{\hbar} [\Delta_c, S] = c_4 s_4 + c_5 s_5 \tag{84}$$

2) the operation

$$\Delta_{\eta_1} = \frac{\hbar}{i} \int dx \, (\eta_1 \delta_{\eta_1})(x) \tag{85}$$

corresponds to a variation of z_1 in the neighbourhood of $z_1 = z_2 = 1$, $z_1' = 0$ and its covariance is given by

$$\frac{i}{\hbar} [\Delta_{\eta_1}, S] = - c_2 s_2 \tag{86}$$

3) the operation

$$\Delta_{\eta_2} = \frac{\hbar}{i} \int dx \, (\eta_2 \delta_{\eta_2})(x) \tag{87}$$

corresponds to a variation of z_2. Its covariance is given by

$$\frac{i}{\hbar} [\Delta_{\eta_2}, S] = - c_1 s_1 \tag{88}$$

4) the operation

$$\Delta'_{\eta_1} = \frac{\hbar}{i} \int dx \, (\eta_1 \delta_\Xi)(x) \tag{89}$$

corresponds to a variation of z_1'. Its covariance is given by

$$\frac{i}{\hbar} [\Delta'_{\eta_1}, S] = - c_2 s_3 \tag{90}$$

It is obvious that all of these four insertions leave all physical normalization conditions (3I, 3, 4, 5, 6, 7, 8) unchanged.

$\hat{\Delta}_\lambda$ is thus a linear combination of $\Delta^0_4, \Delta^0_5, \Delta_{\eta_1}, \Delta_{\eta_2}, \Delta'_{\eta_1}$ which solves part of Eq.(74).

We are thus left with finding a basis of symmetrical insertions.

We know that, given the Slavnov identity $\mathcal{L}^{eff}(\varphi, \eta)$ depends on nine parameters, namely six to specify $\mathcal{L}^{eff}(\varphi)$, three to specify the external field dependence (i.e. z_1, z_2, z_1').

This is indeed true in the tree approximation and therefore, by the theorem of appendix II, to all orders. (Of course, this counting does not take into account any of the normalization conditions (31), including (31,0)). As a consequence, there are nine independent symmetric insertions.

We first construct those which respect the physical normalization conditions :

The first one is :

$$\Delta_0^{0,S} = \Delta_4^0 + \Delta_5^0 - \Delta_c \tag{91}$$

The second one is generated by the variation $\delta\varphi_1 = \text{const.}$:

$$\Delta_1^{0,S} = \int J_1(x)dx \tag{92}$$

The third one is obtained by considering

$$\Delta\varphi_2 = \frac{\hbar}{i}\int dx \left(J_2 \frac{\delta}{\delta J_2}\right)(x) \tag{93}$$

whose covariance is given by

$$\frac{i}{\hbar}\left[\Delta_{\varphi_2}, \delta\right] = c_2\delta_2 + c_3\delta_3 - c_5\delta_5 \tag{94}$$

From the foregoing analysis:

$$\Delta_2^{0,S} = \Delta_{\varphi_2} + \Delta_{\eta_1} + \frac{c_3}{c_2}\Delta_{\eta_1}' + \Delta_5^0 \tag{95}$$

is symmetric, leaves the physical normalization conditions unchanged, and is non zero as can be seen by a direct calculation at the tree level.

We are thus left with finding six independent symmetric insertions. By the general theorem of appendix II, five of them are determined by the terms of $\mathcal{L}$ (excluding the one which leads to $\Delta_0^{0,S}$). The sixth one involves

$$\Delta_A = \frac{\hbar}{i}\int dx \left(J_\mu \frac{\delta}{\delta J_\mu}\right)(x) \tag{96}$$

since

$$\frac{i}{\hbar}\left[\Delta_A, \delta\right] = c_0\delta_0 - c_4\delta_4 = -c_1\delta_1 - c_2\delta_2 - c_3\delta_3 - 2c_4\delta_4 - c_5\delta_5 \tag{97}$$

it follows that

$$\Delta_6^S = \Delta_A - \Delta_{\eta_2} - \Delta_{\eta_1} - \frac{c_3}{c_2}\Delta_{\eta_1}' + 2\Delta_c - \Delta_5^0 \tag{98}$$

is symmetric.

It is straightforward but tedious to verify in the tree approximation that these six insertions alter independently the six physical normalization conditions(31,3,4,5,6,7,8).

The gauge invariance proof is thus completed . It is extended in appendix VII to gauges which contain a quadratic term odd under charge conjugation .

IV- UNITARITY OF THE S OPERATOR

Let us first define

$$S_{phys}(\underline{J}) = \sum_{phys} Z(\underline{J}) \tag{99}$$

where the notations are the same as in Eq.(33). According to the reduction formula , the physical S operator is given by

$$S_{phys} = S_{phys}(\underline{J})\Big|_{\underline{J}=0} \tag{100}$$

The contribution of non physical particle states to physical unitarity is explicitly given in the expression :

$$S^{\dagger}_{phys}(\underline{J}) \exp\left\{ i\hbar \int dx \left[\frac{\overleftarrow{\delta}}{\delta \underline{J}} * \overleftarrow{L}\, S_+ * \overrightarrow{L} \frac{\overrightarrow{\delta}}{\delta \underline{J}} \right](x) \right\} S_{phys}(\underline{J})\Big|_{\underline{J}=0}$$
$$\equiv S^{\dagger}_{phys}(\underline{J}) \exp \mathcal{A}\, S_{phys}(\underline{J})\Big|_{\underline{J}=0} \tag{101}$$

Here L and $i\hbar S_+$ are respectively the differential operator occuring in the asymptotic field equations and the positive frequency part of the asymptotic field commutator.

The proof consists in considering

$$S^{\dagger}_{phys}(\underline{J})\, U(\lambda)\, S_{phys}(\underline{J})\Big|_{\underline{J}=0} = S^{\dagger}_{phys}(J) \exp(\lambda \mathcal{A})\, S_{phys}(\underline{J})\Big|_{J=0} \tag{102}$$

and evaluating

$$\partial_\lambda S^{\dagger}_{phys}(\underline{J})\, U(\lambda)\, S_{phys}(\underline{J})\Big|_{\underline{J}=0} \tag{103}$$

It is shown in appendix V, by extensive use of the Slavnov identity that

$$\partial_\lambda S^{\dagger}_{phys}(\underline{J})\, U(\lambda)\, S_{phys}(\underline{J})\Big|_{\underline{J}=0} =$$

$$= \lim_{\varepsilon\to 0} S^{\dagger}_{phys}(\underline{J}) \left\{ i\hbar \left[\frac{\overleftarrow{\delta}}{\delta J(g)} \left(\overleftarrow{L}\, S_+^{\varepsilon} * \overrightarrow{L} \right) \frac{\overrightarrow{\delta}}{\delta J(g)} \right]_{gg}(x)\, dx \right\} U(\lambda)\, S_{phys}(\underline{J}) \Big|_{\underline{J}=0} \tag{104}$$

where

$$\delta_{J(g)} = \bar{a}\, \partial_\nu \frac{\delta}{\delta J_\nu} + \bar{\rho}\, \frac{\delta}{\delta J_2} \tag{105}$$

S_+^{ε} denotes a regularized version of S_+ around the ghost mass shell and the index gg labels the "gauge-gauge" matrix element of $\overleftarrow{L}\, S_+^{\varepsilon}\, \overrightarrow{L}$ as indicated in appendix V.

Integrating Eq.(104) with respect to λ yields

$$S^{\dagger}_{phys}(\underline{J})\, U(1)\, S_{phys}(\underline{J})\Big|_{\underline{J}=0} = \lim_{\varepsilon\to 0} S^{\dagger}_{phys}(\underline{J}) \exp\left\{ i\hbar \int dx \left[\frac{\overleftarrow{\delta}}{\delta J(g)} \left(\overleftarrow{L}\, S_+^{\varepsilon} * \overrightarrow{L} \right) \frac{\overrightarrow{\delta}}{\delta J(g)} \right]_{gg}(x) \right\} S_{phys}(\underline{J}) \Big|_{\underline{J}=0} \tag{106}$$

Further use of the Slavnov identity according to which , the gauge operator decouples from physical states finishes the proof:

$$S^{\dagger}_{phys}(\underline{J})\, U(1)\, S_{phys}(\underline{J})\Big|_{\underline{J}=0} = S^{\dagger}_{phys}(\underline{J})\, S_{phys}(\underline{J})\Big|_{\underline{J}=0}$$

$$= S^{\dagger}_{phys}\, S_{phys} \,. \tag{107}$$

Unitarity follows from the hermiticity of the Lagrangian

CONCLUSION . OUTLOOK

The gauge invariance problem has been solved for the abelian Higgs Kibble model treated in a family of gauges odd under charge conjugation . Emphasis was put on the fulfillment of normalization conditions which allow the interpretation of the theory within a Fock space with indefinite metric . This has in particular allowed us to prove the unitarity of the physical scattering operator and to construct some physical local observables. We feel however that one should make a more complete study of the zero norm states that are allowed in the definition of physical states as equivalence classes. From the technical point of view , it was encouraging to see that the theory was widely controlled by the algebraic structure of its tree approximation thanks to the repeated application of the implicit function theorem for formal power series . This situation looks quite favorable to a future treatement of the non abelian situations, at least when no fermion anomalies are potentially present. This last case will doubtlessly call for more refined techniques, involving the Callan-Symanzik equations which have not been included here.

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APPENDIX I : NON ABELIAN GAUGE TRANSFORMATIONS: CLASSICAL THEORY

Here are a few details concerning the classical theory of non abelian gauge transformations: the gauge parameters Λ as well as the $\phi\pi$ ghost field $\bar{c}$ are labelled by the indices of the dual of a Lie algebra $\mathcal{F}$ with structure constants f . The $\phi\pi$ ghost field c and the gauge function $\mathcal{G}$ are labelled by the Lie algebra itself . $m = \frac{\delta \mathcal{G}}{\delta \Lambda}$ is labelled as a linear operator from $\mathcal{F}$ into $\mathcal{F}$. The square of $\mathcal{G}$ is the Killing form of $\mathcal{F}$, at least for the non degenerate part.

Going from the Ward identity to the Slavnov identity now involves an extra term:

$$-\int dx\,dy\,dz\, c_\alpha(x) \left(\frac{\delta^2 \mathcal{G}_\alpha(x)}{\delta \Lambda^\gamma_{(z)} \delta \Lambda^\beta_{(y)}} \right) \bar{c}^\gamma_{(z)} \bar{c}^\beta_{(y)} \qquad \text{(AI,I)}$$

which , using the group law together with the anticommutativity of $\bar{c}$ boils down to

$$-\frac{1}{2}\int dx\,dy\, c_\alpha(x) \frac{\delta \mathcal{G}_\alpha(x)}{\delta \Lambda^\delta_{(y)}} f^\delta_{\beta\gamma} \bar{c}^\gamma_{(y)} \bar{c}^\beta_{(y)} \qquad \text{(AI,2)}$$

or , using the equation of motion:

$$\frac{1}{2}\int dx\, \bar{c}^\alpha(x) f^\gamma_{\alpha\beta} \bar{c}^\beta(x) \Xi_\gamma(x) \qquad \text{(AI,3)}$$

The corresponding Slavnov identity can then be interpreted as expressing the invariance of the lagrangian under the transformation law:

$$\delta \varphi_i(x) = \lambda \int dy \frac{\delta \varphi_i(x)}{\delta \Lambda^\alpha(y)} \bar{c}^\alpha(y) \qquad \text{(AI,4)}$$

$$\delta c_\alpha(x) = \lambda\, \mathcal{G}_\alpha(x) \quad , \quad \delta \bar{c}^\alpha(x) = \frac{\lambda}{2} f^\alpha_{\beta\gamma} \bar{c}^\beta(x) \bar{c}^\gamma(x)$$

where λ is a space time independent anticommuting parameter carrying no index.

APPENDIX II. The implicit function theorem for formal power series

This appendix is devoted to the statement and proof of an easy theorem (11) which has been repeatedly used to reduce the proof of a property to all orders of perturbation theory down to the verification of a simple algebraic property of the tree approximation :

Th Let $F_i(x_1,..,x_n ; y_1,..,y_p) = 0 \ (i=1,..,n)$ be a set of algebraic analytic equations which has a unique solution $x_i = \varphi_i(y_1,..,y_p)$ (analytic) in $(y_1,..,y_p)$ in some neighbourhood of $(y_1^0,..,y_p^0)$.
Then the perturbed system

$$F_i(\underline{x}_1,..,\underline{x}_n ; \underline{y}_1,..,\underline{y}_p) = t\, f_i(\underline{x}_1,..,\underline{x}_n ; \underline{y}_1,..,\underline{y}_p ; t) \quad (i=1,..,n)$$

where $\underline{y}_1,..,\underline{y}_p$ are formal power series in t whose lower order terms are $y_1^0,..,y_p^0$ and the f_i's are formal power series in $\underline{x}_1,..,\underline{x}_n, \underline{y}_1,..,\underline{y}_p, t$, possesses a unique solution

$$\underline{x}_i = \underline{\varphi}_i(t, \underline{y}_1,..,\underline{y}_p)$$

where the $\underline{\varphi}_i$'s are formal power series in $t, \underline{y}_1,..,\underline{y}_p$

Proof :

Let $\underline{\xi}_i = \underline{x}_i - \varphi_i(\underline{y}_1,..,\underline{y}_p)$

F_i can be expanded into a formal power series in $\xi_i, \underline{y}_i$, whose term linear in $\underline{\xi}$ is

$$\left.\frac{\partial F_i}{\partial x_j}\right|_{x_k = \varphi_k(\underline{y}_1,..,\underline{y}_p)} \underline{\xi}_j$$

where, by the hypothesis

$$\det \left\| \frac{\partial F_i}{\partial x_j} \right\|$$

is invertible in the sense of formal power series. Hence the initial system can be cast into the form

$$\left.\frac{\partial F_i}{\partial x_j}\right|_{x_k = \varphi_k(\underline{y}_1,..,\underline{y}_p)} \underline{\xi}_j = \phi_i(\underline{\xi}_1,..,\underline{\xi}_n ; \underline{y}_1,..,\underline{y}_p ; t)$$

where the formal power series ϕ_i are such that $\phi_i(0,..,0 ; \underline{y}_1,..,\underline{y}_p ; 0) = 0$

i.e

$$\underline{\xi}_j = \psi_j(\underline{\xi}_1,..,\underline{\xi}_n ; \underline{y}_1,..,\underline{y}_p ; t)$$

with the same conditions on ψ_j. This system is easily solved by iteration.

-APPENDIX III-

We show here that, as a consequence of the Slavnov identity, $\Gamma^{-1}_{\bar{c}c}\left(\Gamma_{A^L A^L}\Gamma_{\varphi_2\varphi_2} - \Gamma^2_{A^L\varphi_2}\right)$ is finite at $p^2 = m_q^2$ and thus can be required to vanish. In other words, the $\phi\pi$ ghost mass is always degenerate with one of the A^L, φ_2 ghost masses. Complete degeneracy then characterizing spontaneous breakdown.

We first write the Slavnov identity in terms of the vertex functional :

$$\mathcal{S}(\Gamma) \equiv \int dx \left(-\bar{c}\,\partial_\mu \frac{\delta\Gamma}{\delta A_\mu} - \bar{e}_1 \frac{\delta\Gamma}{\delta\varphi_1}\frac{\delta\Gamma}{\delta\eta_2} + \bar{e}_2 \frac{\delta\Gamma}{\delta\varphi_2}\frac{\delta\Gamma}{\delta\eta_1} \right.$$

$$\left. + \bar{m}\,\bar{c}\frac{\delta\Gamma}{\delta\varphi_2} - \bar{a} A_\mu \partial_\mu \frac{\delta\Gamma}{\delta c} + \bar{\beta}\,\varphi_2 \frac{\delta\Gamma}{\delta c} \right)(x) = 0 \qquad \text{(AIII, 1)}$$

Within the A_μ, φ_2 channel, we get :

$$i p_\mu \Gamma_{A_\mu\varphi_2}(p) - \bar{\beta}\,\Gamma_{\bar{c}c}(p^2) - \gamma(p^2)\,\Gamma_{\varphi_2\varphi_2}(p^2) = 0$$

$$p_\mu p_\nu \Gamma_{A_\mu A_\nu}(p) - \bar{a}\,p^2 \Gamma_{\bar{c}c}(p^2) + i\,\gamma(p^2)\, p_\mu \Gamma_{A_\mu\varphi_2}(p) = 0 \qquad \text{(AIII,2)}$$

where

$$\gamma(p^2) = \bar{e}_2\,\Gamma_{\bar{e}\eta_1}(p^2) + \bar{m} \qquad \text{(AIII,3)}$$

Thus

$$\Gamma_{\varphi_2\varphi_2}(p^2) = \frac{1}{\gamma(p^2)}\left[i p_\mu \Gamma_{A_\mu\varphi_2}(p) - \bar{\beta}\,\Gamma_{\bar{c}c}(p^2)\right] \qquad \text{(AIII,4)}$$

$$p_\mu p_\nu \Gamma_{A_\mu A_\nu}(p) = i\,\gamma(p^2)\, p_\mu \Gamma_{A_\mu\varphi_2}(p) + \bar{a}\,p^2\,\Gamma_{\bar{c}c}(p^2)$$

Hence :

$$\mathcal{D} = \det\begin{pmatrix} \Gamma_{A^L A^L} & \Gamma_{A^L\varphi_2} \\ \Gamma_{A^L\varphi_2} & \Gamma_{\varphi_2\varphi_2} \end{pmatrix} = \frac{1}{p^2\gamma(p^2)}\left[-\bar{\beta}\bar{a}\,p^2 \left(\Gamma_{\bar{e}c}(p^2)\right)^2 \right. \qquad \text{(AIII,5)}$$

$$\left. + i\, p_\mu \Gamma_{A_\mu\varphi_2}(p)\,\Gamma_{\bar{e}c}(p^2)\left(\bar{a}^2 p^2 - \bar{\beta}\,\gamma(p^2)\right)\right.$$

Thus, first

$$\Gamma^{-1}_{\bar{e}c}\, D\Big|_{p^2=m_q^2} \qquad \text{is finite}$$

and furthermore its vanishing implies

$$a\,p^2 - \bar{\rho}\left(\bar{m} + \bar{e}_2\,\Gamma_{\bar{e}\eta_1}(p^2)\right)\Big|_{p^2=m_q^2} \equiv a\,p^2 - \bar{\rho}\,\gamma(p^2)\Big|_{p^2=m_q^2} = 0 \qquad \text{(AIII, 6)}$$

since $\Gamma \neq 0$, provided one stays away from the restricted' t Hooft gauge $(\rho = \alpha e v)$. A_μ, φ_2 .

Looking now at the $\phi\pi$ propagator equation

$$\left[\bar{a}\,p^2 - \left(\bar{\rho}\,\bar{w} + \bar{\mu}^2\right)\right] G_{\bar{e}c}(p^2) - \bar{\rho}\,\bar{e}_2\, G_{\bar{e}\eta_1} = \bar{\alpha} \qquad \text{(AIII,7)}$$

the absence of a pole in the left hand side at $p^2 = m_q^2$ yields:

$$\bar{a}\,p^2 - \left(\bar{\rho}\,\bar{w} + \bar{\mu}^2\right) - \bar{\rho}\,\bar{e}_2\,\Gamma_{\bar{e}\eta_1}(p^2)\Big|_{p^2=m_q^2} = 0 \qquad \text{(AIII,8)}$$

after multiplication of (AIII,7) through $\Gamma_{\bar{e}c}$ and use of

$$G_{\bar{e}\eta_1} = \Gamma_{\bar{e}\eta_1}\, G_{\bar{e}c} \qquad \text{(AIII,9)}$$

Hence comparing with Eq.(AIII,6) , we get:

$$\bar{\mu} = 0 \qquad \text{(AIII,IO)}$$

- APPENDIX IV -

We have shown in chapter III (Eq(8I)) that, in the limit $\varepsilon \to 0$, the commutator of $\Delta_{i,\varepsilon}$ with δ is equal to $e_i \delta_i$. We thus infer that the infinite part of $\Delta_{i,\varepsilon}$ as given by the Zimmermann-Wilson expansion is a symmetric insertion with coefficients going to infinity as $\varepsilon \to 0$. The finite part Δ_i will then be given by $[\delta, Q_i]$ where Q_i is the finite part of $Q_{i,\varepsilon}$

It is possible but lengthy to verify these statements by looking at the Zimmermann-Wilson expansion of $Q_{i,\varepsilon}$. In the case of $Q_{5,\varepsilon}$ the calculation is however reasonably simple:

$$\int dx \left[T \varphi_2(\varepsilon)\, c(-\varepsilon) \right](x) = \frac{i}{\hbar} \Big\langle \int dx\, T\left[\varphi_2(\varepsilon)\, c(-\varepsilon)\right](x)\, N_2[\bar{c}\varphi_2](0) \Big\rangle \int dx\, \eta_2(x)$$

$$+ \Big\langle \int dx\, T\left[\varphi_2(\varepsilon)\, c(-\varepsilon)\right](x)\, \tilde{\varphi}_2(0)\, \bar{c}(0) \Big\rangle \cdot \int dx \Big\{ T\, N_2[\varphi_1(\varepsilon)\, c(-\varepsilon)](x) \Big\} \qquad \text{(AIV,1)}$$

where the second coefficient is amputated on its $\tilde{\varphi}_2$ $\bar{c}$ arguments.

The only singular coefficient in this expansion is

$$\Big\langle \int dx\, T\left[\varphi_2(\varepsilon)\, c(-\varepsilon)\right](x)\, N_2[\bar{c}\varphi_2](0) \Big\rangle \qquad \text{(AVI2)}$$

which diverges logarithmically . The singular part of $\Delta_{5,\varepsilon}$ is thus proportional to $\left[\delta, \int dx\, \eta_2(x)\right] \sim \int dx\, J_1(x)$ which is symmetrical (cf. (92)) as expected .

By a similar but more involved analysis one can evaluate the singular part of $\Delta_{4,\varepsilon}$ which assumes the form

$$\omega(\varepsilon) \int dx\, J_1(x) + \omega'(\varepsilon) \Big\{ \int dx \left(J_1 \frac{\delta}{\delta J_1} + \eta_2 \frac{\delta}{\delta \eta_2} + J_2 \frac{\delta}{\delta J_2} + \eta_1 \frac{\delta}{\delta \eta_1} + \frac{\bar{m}}{\bar{e}_2} \eta_1 \frac{\delta}{\delta \xi} \right)(x) \qquad \text{(AIV, 3)}$$

$$+ \Delta_5 \Big\}$$

where ω and ω' are , in the limit $\varepsilon \to 0$ logarithmically divergent .

The resulting finite parts are however not suitable for our purpose because , due to the occurence of graphs which are φ_1 one particle reducible they do not vanish upon application of the operator $\Sigma_{\text{1 plys}}$ (cf.

Since the $Q_{i,\Sigma}$'s carry the quantum numbers of $\bar{c}\varphi_2$ we have:

$$\sum_{phys} \delta Q_i Z = -\bar{e}_1 \sum_{phys} \int dx \, (J_1 \delta_{\eta_2})(x) \, Q_i Z$$

$$= -\bar{e}_1 \sum_{phys} \int dp \, \tilde{J}_1(p) \, \Gamma^{(Q_i)}(p) \, \delta_{\tilde{J}_1(p)} Z \qquad \text{(AIV,4)}$$

$$= -\bar{e}_1 \, \Gamma^{(Q_i)}_{(M^2)} \sum_{phys} \int dp \, \tilde{J}_1(p) \, \delta_{\tilde{J}_1(p)} Z$$

$$= -\bar{e}_1 \, \Gamma^{(Q_i)}_{(M^2)} \sum_{phys} \int dx \, (J_1 \delta_{J_1} + \eta_2 \delta_{\eta_2})(x) Z$$

where

$$\Gamma^{(Q_i)}(p) = \frac{i}{\hbar} \langle T \, N_2[\bar{c}\varphi_2](0) \, Q_i \, \tilde{\underline{\varphi}}_1(p) \rangle \qquad \text{(AIV,5)}$$

is involved in the expansion:

$$\delta_{\tilde{\eta}_2(p)} Q_i Z = \frac{i}{\hbar} \langle T N_2[\bar{c}\varphi_2](0) Q_i \tilde{\varphi}_1(p) \rangle \delta_{\tilde{J}_1(p)} Z + \frac{i}{\hbar} \langle T N_2[\bar{c}\varphi_2](p) Q_i \rangle^{I_1} Z \qquad \text{(AIV,6)}$$

where the upperscript I_1 denotes the set of graphs which are one particle irreducible with respect to the pair $\bar{c}\varphi_2, Q_i$.

Since $\int dx \, [J_1 \delta_{J_1} + \eta_2 \delta_{\eta_2}](x)$ is obviously a symmetric insertion, adding $\bar{e}_1 \, \Gamma^{(Q_i)}_{(M^2)} \int dx \, [J_1 \delta_{J_1} + \eta_2 \delta_{\eta_2}](x)$ does not change the covariance of $[\delta, Q_i]$ and produces insertions which leave the physical normalization conditions (Eq.31(3,4,5,6,7)) invariant.

We now want to show that the insertions Δ^0_i leave the normalization condition Eq.(31,8) unchanged. These insertions can be replaced by $\Delta_i = [\delta, Q_i]$ modulo terms which trivially do not contribute to the calculation.

We shall show that:

$$\Gamma^{-2}_{\bar{c}c} \det \begin{pmatrix} \Gamma_{A^L A^L} & \Gamma_{A^L \varphi_2} \\ \Gamma_{A^L \varphi_2} & \Gamma_{\varphi_2 \varphi_2} \end{pmatrix} \equiv G^2_{c\bar{c}} D \qquad \text{(AIV,7)}$$

remains regular at the $\phi\pi$ mass, upon insertion of Δ_i.

$$\Delta_i (G^2_{c\bar{c}} D) = G_{c\bar{c}} D \left(2[\Delta_i G_{c\bar{c}}] \right.$$

$$\left. - G_{c\bar{c}} \, T_2 \left\{ \begin{pmatrix} \Gamma_{A^L A^L} & \Gamma_{A^L \varphi_2} \\ \Gamma_{A^L \varphi_2} & \Gamma_{\varphi_2 \varphi_2} \end{pmatrix} \left[\Delta_i \begin{pmatrix} G_{A^L A^L} & G_{A^L \varphi_2} \\ G_{\varphi_2 A_L} & G_{\varphi_2 \varphi_2} \end{pmatrix} \right] \right\} \right) \qquad \text{(AIV,8)}$$

where the matrix $\| G_{ij} \|$ is the inverse of the matrix $\| \Gamma_{ij} \|$.
By commuting $\delta_\xi \delta_{\bar\xi}$ through δ we get :

$$\left[\Delta_i G_{c\bar c}\right](p) = \langle T\, \underline{\tilde{\bar c}}(p)\, \mathcal{G}(0)\, Q_i \rangle$$
$$= G_{c\bar c}(p^2) \langle T\, \underline{\tilde{\bar c}}(p)\, \mathcal{G}(0)\, Q_i \rangle \qquad \text{(AIV,9)}$$

where underlining means amputation.

Similarly

$$\left[\Delta_i G^2_{c\bar c} D\right](p) = 2 G^2_{c\bar c}(p^2) D(p^2) \Big[\langle T\, \underline{\tilde{\bar e}}(p)\, \mathcal{G}(0)\, Q_i \rangle$$
$$+ \langle T\, \underline{\tilde{\varphi}_2}(p) \left[\bar e_2 N_2[\bar c \varphi_1] + \bar m \bar e\right](0)\, Q_i \rangle$$
$$- p^2 \langle T\, \underline{\tilde{\partial_\mu A_\mu}}(p)\, \bar c(0)\, Q_i \rangle \qquad \text{(AIV,10)}$$

Thus we only have to make sure that the bracket is regular at the $\phi\pi$ squared mass. Now the first term is singular due to the occurence of the $\mathcal{G}$ propagator, and the last two terms are singular due the occurence of the $\phi\pi$ propagators. The $\mathcal{G}$ propagator can be factorized using the Slavnov identities:

$$\langle T\, \tilde{\mathcal{G}}(p)\, \varphi_2(0) \rangle = -\langle T \left(\bar e_2 N_2[\bar c \varphi_1] + \bar m \bar e\right)(0)\, \tilde c(p) \rangle$$
$$\langle T\, \tilde{\mathcal{G}}(p)\, \partial_\mu A_\mu(0) \rangle = p^2 \langle T\, \bar c(0)\, \tilde{\bar e}(p) \rangle \qquad \text{(AIV,11)}$$

so that:

$$\langle T\, \underline{\tilde{\bar e}}(p)\, \mathcal{G}(0) Q_i \rangle = -\langle T\, \underline{\tilde{\bar e}}(p) \varphi_2(0)\, Q_i \rangle \langle T \left(\bar e_2 N_2[\bar c \varphi_1] + \bar m \bar e\right)(0)\, \tilde c(p) \rangle$$
$$+ p^2 \langle T\, \underline{\tilde{\bar e}}(p)\, \partial_\mu A_\mu(0) Q_i \rangle \langle T\, \bar e(0)\, \tilde c(p) \rangle \qquad \text{(AIV,12)}$$

Now the $\phi\pi$ equation of motion allows to replace $[\bar{e}_2 N_2[\bar{e}\varphi_1] + \bar{m}\bar{c}](0)$ by a term proportional to $\Box\,\bar{c}(0)$ up to a regular term, so that the factors $\langle T\,\bar{c}(0)\,\tilde{c}(p)\rangle$ undo the $\bar{c}$ amputation involved in their factors and produce an exact cancellation with the last two terms in Eq.(AIV,10).

- APPENDIX V -

This appendix deals with a number of details in the unitarity proof of ch. IV. We first discuss the properties of the asymptotic ghost field wave operators L and of the corresponding asymptotic field two point function S_+.

Within the coupled $(\partial_\mu A_\mu, \varphi_2)$ channels L can be taken as a polynominial approximation to the matrix

$$\begin{pmatrix} \Gamma_{gg} & \Gamma_{\bar{g}g} \\ \Gamma_{\bar{g}g} & \Gamma_{\bar{g}\bar{g}} \end{pmatrix}$$

where

$$\begin{aligned} g &= \bar{\alpha}\,\partial_\mu A_\mu + \bar{\beta}\varphi_2 \\ \bar{g} &= -\bar{\alpha}\,\partial_\mu A_\mu + \bar{\beta}\varphi_2 \end{aligned} \qquad \text{(AV,1)}$$

Denoting $x = p^2 - m_4^2$, and taking into account: (i) the normalization condition (31,8), which implies the occurence of a double zero in $\det L$ at $x=0$ (ii) the lack of singularity in the gg propagator, which follows from the Slavnov identity and implies the occurence of a double zero in $\Gamma_{\bar{g}g}$ at $x=0$, We can parametrize L in the following form:

$$L = \begin{pmatrix} A+\beta x & \gamma x \\ \gamma x & 0 \end{pmatrix} + O(x^2) \qquad \text{(AV,2)}$$

The last term giving corrections of order x^2.

The corresponding matrix propagator is:

$$\begin{pmatrix} G_{gg} & G_{g\bar{g}} \\ G_{g\bar{g}} & G_{\bar{g}\bar{g}} \end{pmatrix} = -\frac{1}{\gamma^2 x^2}\begin{pmatrix} 0 & -\gamma x \\ -\gamma x & A+\beta x \end{pmatrix} + \text{Regular terms} \qquad \text{(AV,3)}$$

and the S_+ operator is given by:

$$\widetilde{it\,S_+} = \theta(p_0)\begin{pmatrix} 0 & \frac{1}{\gamma}\delta(x) \\ \frac{1}{\gamma}\delta x & -\frac{\beta}{\gamma^2}\delta(x) + \frac{A}{\gamma^2}\delta'(x) \end{pmatrix} \qquad \text{(AV,4)}$$

so that

$$it\left(\overleftarrow{L}\,S_+\,\vec{L}\right)_g = \theta(p_0)\begin{pmatrix} A[\overleftarrow{x}\,\delta(x) + \delta(x)\vec{x} + \overleftarrow{x}\,\delta'(x)\vec{x}] + \beta\overleftarrow{x}\,\delta(x)\vec{x} & \gamma\overleftarrow{x}\,\delta(x)\vec{x} \\ \gamma\overleftarrow{x}\,\delta(x)\vec{x} & 0 \end{pmatrix} \qquad \text{(AV,5)}$$

where the symbols $\vec{x}, \overleftarrow{x}$ are to remind that the usual identities $x\delta(x) = x^2\delta'(x) = 0$ cannot be used here because this kernel is to be tested with functions which have poles at x=0.

Concerning the Faddeev-Popov fields, let us define

$$\begin{aligned} \varphi &= c \\ \bar{\varphi} &= 2\bar{\beta}\left[\bar{m} + \bar{\varepsilon}_2 \Gamma_{\bar{c}\eta}(m_q^2)\right]\bar{c} \end{aligned} \tag{AV,6}$$

Using the results of Appendix III we get:

$$G_{\varphi\bar{\varphi}} = \frac{1}{\gamma x} + \text{regular Terms} \tag{AV,7}$$

and

$$i\hbar\,(\overleftarrow{L}\, S_+ \vec{L})_{\phi\bar{\eta}} = \gamma\, \overleftarrow{x}\, \delta(x)\, \vec{x} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{AV,8}$$

The $\mathcal{A}$ operator Eq(101) can now be written

$$\begin{aligned} \mathcal{A} = \int dp\, \theta(p) \Big\{ & \vec{\delta}_{\tilde{J}_{\bar{q}}(-p)} \left[A\left(\overleftarrow{x}\,\delta(x) + \delta(x)\vec{x} + \overleftarrow{x}\,\delta'(x)\vec{x}\right) + \beta\, \overleftarrow{x}\,\delta(x)\vec{x} \right] \vec{\delta}_{\tilde{J}_q(p)} \\ & + \gamma\, \overleftarrow{x}\,\delta(x)\vec{x} \Big[\vec{\delta}_{\tilde{J}_{\bar{q}}(-p)} \vec{\delta}_{\tilde{J}_q(p)} + \overleftarrow{\delta}_{\tilde{J}_q(p)} \vec{\delta}_{\tilde{J}_{\bar{q}}(p)} + \overleftarrow{\delta}_{\tilde{J}^*_{\bar{c}}(-p)} \vec{\delta}_{\tilde{J}_{\bar{c}}(p)} \\ & - \overleftarrow{\delta}_{\tilde{J}^*_{\bar{c}}(-p)} \vec{\delta}_{\tilde{J}_{\bar{c}}(p)} \Big] \Big\}_{x = p^2 - m_q^2} \end{aligned} \tag{AV,9}$$

where J_φ is the source of the field $\varphi^\dagger$ that is used in the definition of the antitime ordered functional .

Before pursuing, let us regularize the δ functions according to

$$\delta(x) \to \delta_\varepsilon(x) = \frac{e^{-\frac{x^2}{\varepsilon}}}{\sqrt{\pi\varepsilon}} \tag{AV,10}$$

so that we may forget about the arrows on the $\vec{x}$ variables.

Owing to the invariance of the lagrangian under the transformation:

$$\begin{aligned} c &\to -\bar{c} \\ \bar{c} &\to c \end{aligned} \tag{AV,11}$$

we have the identity

$$\overleftarrow{\delta}_{\tilde{J}^*_{\bar{c}}(-p)} \vec{\delta}_{\tilde{J}_{\bar{c}}(p)} - \overleftarrow{\delta}_{\tilde{J}^*_{\bar{c}}(p)} \vec{\delta}_{\tilde{J}_{\bar{c}}(p)} = \overleftarrow{\delta}_{\tilde{J}^*_{c}(-p)} \vec{\delta}_{\tilde{J}_{\bar{c}}(p)} + \overleftarrow{\delta}_{\tilde{J}^*_{\bar{c}}(-p)} \vec{\delta}_{\tilde{J}_{c}(p)} \tag{AV,12}$$

Taking into account (Eq(43) we get:

$$\begin{aligned} \left[\vec{\delta}_{\tilde{J}_{\bar{q}}(p)}, \vec{S} \right] &= \delta_{\tilde{J}_{\bar{c}}(p)} + O(x) \\ \left[\vec{\delta}_{\tilde{J}_q(p)}, \vec{S} \right] &= -\frac{\bar{\alpha}}{\gamma} x\, \delta_{\tilde{J}_{\bar{c}}(p)} + O(x^2) \\ \left\{ \vec{\delta}_{\tilde{J}_{\bar{c}}(p)}, \vec{S} \right\} &= \delta_{\tilde{J}_q(p)} \\ \left\{ \vec{\delta}_{\tilde{J}_{\bar{c}}(p)}, \vec{S} \right\} &= 0 \end{aligned} \tag{AV,13}$$

which yields

$$[\mathcal{A}, \vec{\mathcal{S}}] = \int dp\, \theta(p_0) \Big\{ \overleftarrow{\delta}_{J_{\bar{\mathcal{G}}}(-p)} \big[2A\alpha\delta(x) + A\alpha^2\delta'(x) + \beta\alpha^2\delta(x) \big] \big(-\tfrac{\bar{\alpha}}{\gamma} x\big) \vec{\delta}_{J_{\mathcal{G}}(p)} + \gamma\alpha^2\delta(x) \Big[\overleftarrow{\delta}_{J_{\bar{\mathcal{G}}}(-p)} \big(-\tfrac{\bar{\alpha}}{\gamma} x\big) \vec{\delta}_{J_{\bar{\mathcal{E}}}(p)} + \overleftarrow{\delta}_{J_{\mathcal{G}}(-p)} \vec{\delta}_{J_{\bar{\mathcal{E}}}(p)} + \overleftarrow{\delta}_{J^*_{\bar{\mathcal{E}}}(-p)} \vec{\delta}_{J_{\mathcal{G}}(p)} \Big] \Big\} \qquad \text{(AV,14)}$$

$$+ O(\varepsilon)$$

Since the propagator attached to the $\mathcal{G}$ and $\bar{\mathcal{E}}$ legs have only simple poles the β dependent term in the righthand side of (Eq.(AV,14)) is of order ε. This does not happen for the term involving $\overleftarrow{\delta}_{J_{\bar{\mathcal{G}}}}$ because the $(\bar{\mathcal{G}}, \mathcal{G})$ propagator has a double pole.

However we have

$$\times G_{\bar{\mathcal{G}}\bar{\mathcal{G}}}(p^2) = -\frac{A}{\gamma} G_{\mathcal{G}\bar{\mathcal{G}}}(p^2) + \text{Regular Terms.} \qquad \text{(AV,15)}$$

Since the $(\mathcal{G}, \mathcal{G})$ propagator has no pole we have:

$$\times \vec{\delta}_{J_{\bar{\mathcal{G}}}(p)} Z = -\frac{A}{\gamma} \vec{\delta}_{J_{\mathcal{G}}(p)} Z \qquad \text{+Regular terms} \qquad \text{(AV,16)}$$

Thus

$$[\mathcal{A}, \vec{\mathcal{S}}] = \int dp\, \theta(p_0) \Big\{ \overleftarrow{\delta}_{J_{\bar{\mathcal{G}}}(-p)} \big[2A\alpha\delta(x) + A\alpha^2\delta'(x) - A\alpha\delta(x) \big] \big(-\tfrac{\bar{\alpha}}{\gamma} x\big) \vec{\delta}_{J_{\bar{\mathcal{E}}}(p)} + \gamma\alpha^2\delta(x) \Big[\overleftarrow{\delta}_{J_{\bar{\mathcal{G}}}(-p)} \vec{\delta}_{J_{\bar{\mathcal{E}}}(p)} + \overleftarrow{\delta}_{J^*_{\bar{\mathcal{E}}}(-p)} \vec{\delta}_{J_{\mathcal{G}}(p)} \Big] \Big\} + O(\varepsilon)$$

$$= \int dp\, \theta(p_0) \Big\{ \gamma\alpha^2\delta(x) \Big[\overleftarrow{\delta}_{J_{\mathcal{G}}(-p)} \vec{\delta}_{J_{\bar{\mathcal{E}}}(p)} + \overleftarrow{\delta}_{J^*_{\bar{\mathcal{E}}}(-p)} \vec{\delta}_{J_{\mathcal{G}}(p)} \Big] \Big\} + O(\varepsilon) \qquad \text{(AV,17)}$$

The vanishing of the A dependent terms is due to the absence of double poles in the $\mathcal{G}$ and $\bar{\mathcal{E}}$ propagators.

As a consequence we get:

$$[\mathcal{A}, \vec{\mathcal{S}}] = [\overleftarrow{\mathcal{S}}^*, \mathcal{A}] + O(\varepsilon) \qquad \text{(AV,18)}$$

where $\overleftarrow{\mathcal{S}}^*$ is the Slavnov operator which characterizes the anti-time ordered functional.

Indeed Eq.(AV,18) is a consequence of the symmetry of both $\mathcal{A}$ and $[\mathcal{A}, \vec{\mathcal{S}}]$ with respect to the transposition and the complex conjugation of the sources.

We are now in condition to state the following identity:

$$S^{\dagger}_{phys}(\underline{J})\Big\{\int dp\,\theta(p_0)\,\gamma x^2\delta(k)\Big[\overleftarrow{\delta}_{\tilde{J}_{\xi}(-p)}\overrightarrow{\delta}_{\tilde{J}_{\bar{\xi}}(p)}+\overleftarrow{\delta}_{\tilde{J}_{\bar{\xi}}(-p)}\overrightarrow{\delta}_{\tilde{J}_{\xi}(p)}$$
$$+\overleftarrow{\delta}_{\tilde{J}^{*}_{e}(-p)}\overrightarrow{\delta}_{\tilde{J}_{e}(p)}+\overleftarrow{\delta}_{\tilde{J}^{*}_{\bar{e}}(-p)}\overrightarrow{\delta}_{\tilde{J}_{\bar{e}}(p)}\Big]\Big\}e^{\lambda A}\,S_{phys}(\underline{I})\Big|_{\underline{J}=0}$$

$$=S^{\dagger}_{phys}(\underline{J})\Big\{\int dp\,\theta(p_0)\,\gamma x^2\delta(k)\Big[\Big\{\overleftarrow{\delta}_{\tilde{J}^{*}_{e}(-p)},\overleftarrow{S}^{*}\Big\}\overrightarrow{\delta}_{\tilde{J}_{\bar{\xi}}(p)}$$
$$+\overleftarrow{\delta}_{\tilde{J}_{\bar{\xi}}(-p)}\Big\{\overrightarrow{\delta}_{\tilde{J}_{e}(p)},\overrightarrow{S}\Big\}+\Big[\overleftarrow{S}^{*},\overleftarrow{\delta}_{\tilde{J}_{\bar{\xi}}(-p)}\Big]\overrightarrow{\delta}_{\tilde{J}_{e}(p)}$$
$$+\overleftarrow{\delta}_{\tilde{J}^{*}_{e}(-p)}\Big[\overrightarrow{\delta}_{\tilde{J}_{\bar{\xi}}(p)},\overrightarrow{S}\Big]\Big]+O(\varepsilon)\Big\}e^{\lambda A}\,S_{phys}(\underline{I})\Big|_{\underline{I}=0} \qquad \text{(AV,19)}$$

$$=S^{\dagger}_{phys}(\underline{J})\Big\{\int dp\,\theta(p_0)\,\gamma x^2\delta(k)\Big[\overleftarrow{\delta}_{\tilde{J}^{*}_{e}(-p)}\Big[\overleftarrow{S}^{*},e^{\lambda b}\Big]\overrightarrow{\delta}_{\tilde{J}_{\bar{\xi}}(p)}$$
$$-\overleftarrow{\delta}_{\tilde{J}_{\bar{\xi}}(-p)}\Big[\overleftarrow{S}^{*},e^{\lambda b}\Big]\overrightarrow{\delta}_{\tilde{J}_{e}(p)}+\overleftarrow{\delta}_{\tilde{J}_{\bar{\xi}}(-p)}\Big[e^{\lambda b},\overrightarrow{S}\Big]\overrightarrow{\delta}_{\tilde{J}_{\bar{e}}(p)}$$
$$-\overleftarrow{\delta}_{\tilde{J}^{*}_{e}(-p)}\Big[e^{\lambda b},\overrightarrow{S}\Big]\overrightarrow{\delta}_{\tilde{J}_{\bar{\xi}}(p)}\Big]+O(\varepsilon)\Big\}S_{phys}(\underline{J})\Big|_{\underline{J}=0}=O(\varepsilon)$$

In the first step of this reduction, the $O(\varepsilon)$ term takes into account contributions of the kind:

$$S^{\dagger}_{phys}(\underline{J})\int dp\,\theta(p_0)\,\gamma x^3\delta(x)\,\overleftarrow{\delta}_{\tilde{J}^{*}_{\bar{e}}(-p)}\overrightarrow{\delta}_{\tilde{J}_{e}(p)}e^{\lambda b}\,S_{phys}(\underline{J})\Big|_{\underline{J}=0}$$

The second step makes use of the Slavnov identities

$$\overrightarrow{S}\,S_{phys}(\underline{J})=0$$
$$S^{\dagger}_{phys}(\underline{J})\,\overleftarrow{S}^{*}=0 \qquad \text{(AV,20)}$$

and takes advantage of the zero source condition by commuting $\overrightarrow{S}$ to the left and $\overleftarrow{S}^{*}$ to the right.

The last step is a trivial consequence of Eq.(AV,18). Going back to the expression for $\mathcal{A}$, and taking into account the symmetry property, one gets Eq.(104) of section IV.

- APPENDIX VI-

A CLASS OF LOCAL GAUGE INVARIANT OPERATORS

In order to define a local operator $\mathcal{H}(x)$ of dimension d, we shall first consider an effective lagrangian .

$$\mathcal{L}_{eff}^{(\omega)}(x) = \mathcal{L}_{eff}(\underline{\varphi}, \underline{J}, \underline{\eta})(x) + \omega(x)\, N_d[\mathcal{H}(x)] \qquad \text{(AVI, 1)}$$

where ω is a classical field of dimension 4-d.
The first criterion for gauge invariance is

$$\mathcal{S} Z^c(\underline{J}, \underline{\eta}, \omega) = O(\omega^2) \qquad \text{(AVI, 2)}$$

where $\mathcal{S}$ is the operator defined in Eq(43) . Assuming that Eq.(AVI,2) has solutions , they are in one to one correspondence with those found at the tree level . They will be further specified by as many physical normalization conditions as are necessary to specify physical operators of this type at the tree level (namely modulo the ideal generated by operators which vanish on the physical subspace) . It follows that $\mathcal{H}$ is ambiguous up to a linear combination of operators whose tree approximations vanish on the physical subspace .

The proof of gauge invariance then proceeds as usual (ch.III) Keeping terms of the first order in ω, one looks for the most general solution of

$$\partial_\lambda Z(\underline{J}, \underline{\eta}, \omega) = \frac{i}{\hbar} \Delta_\lambda^{(\omega)} Z(\underline{J}, \underline{\eta}, \omega) \qquad \text{(AVI, 3)}$$

which is of the form

$$\Delta_\lambda^{(\omega)} = \Delta_\lambda + \int dx\, \omega(x) \left[\sum_i f_i\, \mathcal{H}_i^S(x) \right] \qquad \text{(AVI, 4)}$$

where the second term in the right hand side of Eq(AVI,4) is a perturbation of the ω dependent part of the solutions of Eq.(AVI,2) in the tree approximation Testing now Eq.(AVI,4) with the physical normalization conditions which specify

$\mathcal{H}(x)$ shows that the problem reduces to check that the perturbations of operators which have nul physical restriction in the tree approximation retain this property to all orders .

Finally the stability of the physical subspace under application of $\mathcal{H}(x)$, up to the zero norm states is a consequence of Eq.(AVI, 2) as follows from a slight generalization of the argument in Ch.IV: defining $S(\omega)$ by replacing $Z(\underline{J},\underline{\eta})$ by $Z(\underline{J},\underline{\eta},\omega)$ in the LSZ definition of S in the overall Fock space, $\mathcal{H}(x)$ is defined according to

$$\mathcal{H}(x) = \frac{\hbar}{i} S^{-1} \frac{\delta S}{\delta \omega(x)}\Big|_{\omega=0}$$
$$\mathcal{H}^{\dagger}(x) = -\frac{\hbar}{i} \frac{\delta S^{\dagger}}{\delta \omega(x)} S\Big|_{\omega=0} \qquad \text{(AVI, 5)}$$

Let E_0 be the projector on the physical subspace generated by φ_λ and A^T_μ quanta . One wishes to show that

$$E_0 \mathcal{H}^{\dagger}(x) E_0 \mathcal{H}(y) E_0 = E_0 \mathcal{H}^{\dagger}(x) \mathcal{H}(y) E_0 \qquad \text{(AVI, 6)}$$

i.e

$$E_0 \frac{\delta S^{\dagger}}{\delta \omega(x)} \frac{\delta S}{\delta \omega(y)} E_0\Big|_{\omega=0} = E_0 \frac{\delta S^{\dagger}}{\delta \omega(x)} S E_0 S^{\dagger} \frac{\delta S}{\delta \omega(y)} E_0\Big|_{\omega=0} \qquad \text{(AVI, 7)}$$

where the unitarity of S has been used .

Eq.(AVI, 7) follows simply from

$$E_0 S E_0 S^{\dagger} E_0 = E_0 \qquad \text{(AVI, 8)}$$

which is the result of Ch.IV.

and from the identities

$$E_0 \frac{\delta S^{\dagger}}{\delta \omega(x)} \frac{\delta S}{\delta \omega(y)} E_0\Big|_{\omega=0} = E_0 \frac{\delta S^{\dagger}}{\delta \omega(x)} E_0 \frac{\delta S}{\delta \omega(y)} E_0\Big|_{\omega=0}$$
$$E_0 \frac{\delta S^{\dagger}}{\delta \omega(x)} S E_0\Big|_{\omega=0} = E_0 \frac{\delta S^{\dagger}}{\delta \omega(k)} E_0 S E_0\Big|_{\omega=0} \qquad \text{(AVI, 9)}$$

which are consequences of the first criterion for gauge invariance :

$$\left[\frac{\delta}{\delta\omega}, s\right] = 0 \qquad \text{(AVI,10)}$$

and of the argument in Ch. IV .

Example

a) d=2 C= +1.

Θ is a linear combination of $\{\varphi_1, \varphi_1^2, \varphi_2^2, A_\mu A_\mu, \bar{c}c\}$

$s\Theta$ is a linear combination of $\{\varphi_2\bar{c}, \varphi_1\varphi_2\bar{c}, \bar{c}\partial A, \bar{c}cA\}$; so is the term $\Theta(\omega)$ in $sZ(J,\eta,\omega)$.

Thus there is no anomaly i.e. there exists one invariant local operator which is a perturbation of

$$\varphi_1^2 + \varphi_2^2 + 2v\varphi_1 \quad ,$$

which is non zero in the physical subspace . This operator is completely determined by e.g.

$$\langle\Omega, \Theta(x)\varphi_{1,in}(y)\Omega\rangle = i\Delta_M^+(x-y)$$

and can serve as a gauge invariant interpolating field operator for $\varphi_{1,in}$.

b) d=3 C=-1 vector operator

It is trivial that $\partial_\mu G_{\mu\nu}$ solves the problem :

$$\left[(\Box g_{\mu\nu} - \partial_\mu\partial_\nu)\delta_{J_\nu(x)}, s\right] = 0$$

$$\partial_\lambda \prod_1^n{}_i (\Box g_{\mu_i\nu_i} - \partial_{\mu_i}\partial_{\nu_i})\delta_{J_{\nu_i}(x_i)} \sum_{\text{plugs}} Z\Big|_{J=\eta=0} = 0 \qquad (\lambda = \alpha, m_G^2)$$

Appendix VII

This appendix is devoted to the main steps involved in the treatement of an extended class of gauges involving quadratic terms odd under charge conjugation . In the tree approximation , the Slavnov transformation (cf Eq(24)) is now taken to be:

$$\begin{aligned}
\delta A_\mu &= \lambda \partial_\mu \bar{c} \\
\delta \varphi_1 &= -\lambda e \varphi_2 \bar{c} \\
\delta \varphi_2 &= \lambda e (\varphi_1 + v) \bar{c} \\
\delta c &= \lambda (\partial_\mu A_\mu + \rho \varphi_2 + \sigma \varphi_1 \varphi_2) = \lambda g \\
\delta \bar{c} &= 0
\end{aligned} \tag{AVII,1}$$

The most general lagrangian fulfilling the corresponding Slavnov identity is now:

$$\begin{aligned}
\mathcal{L} &= -\frac{Z_4}{4} G_{\mu\nu} G_{\mu\nu} + Z_1 (D_\mu \varphi)^* D_\mu \varphi + \mu^2 \varphi^* \varphi - g (\varphi^* \varphi)^2 \\
&- \frac{1}{\alpha} \left[\frac{g^2}{2} + \bar{c} \frac{\delta g}{\delta \Lambda} c \right] + \beta \left[\frac{A_\mu A_\mu}{2} - \bar{c} c + \frac{\sigma}{e} \frac{\varphi_1^2}{2} + \frac{\rho}{e} \varphi_1 \right]
\end{aligned} \tag{AVII,2}$$

where now

$$\frac{\delta g(x)}{\delta \Lambda(y)} = M(x,y) = \left[\Box + \rho e v + e(\rho + \sigma v) + e\sigma(\varphi_1^2 - \varphi_2^2) \right](x)\, \delta(x-y) \tag{AVII,3}$$

Keeping the normalization conditions (Eq.(31) unchanged Eq.(32) is unchanged except for Eq.(32,3) which now reads

$$M^2 = \frac{2}{Z_1} \left(6 g v^2 - \mu^2 - \frac{\beta \sigma}{2e} \right) \tag{AVII,4}$$

But due to Eq.(32,8) $(\beta = 0)$, the overall algebraic system Eq.(32) is unchanged . We now turn to the details of the Slavnov identity which we shall express in linear form as in Eq.(43). Before doing so we need to introduce at least one external field γ coupled to $\varphi_1 \varphi_2$, to which we assign dimension two and odd charge conjugation quantum number . The corresponding term however undergoes a variation under the Slavnov transformation (AVII,1) , which forces us also to introduce at least one field coupled to $(\varphi_1^2 - \varphi_2^2)\bar{c}$. However , for later use , we shall right away introduce three fields of dimension one , $\vec{\beta} \equiv (\beta_1, \beta_2, \beta_3)$

coupled to three independent linear combinations of : $\vec{\bar{\varphi}} \equiv (\bar{\varphi}_1, \bar{\varphi}_2, \bar{\varphi}_3)$

$\equiv (\varphi_1^2 \bar{c},\ \varphi_2^2 \bar{c},\ A_\mu A_\mu \bar{c})$ and also a field of dimension dimension zero $\varkappa$ coupled to $A_\mu \bar{c} \partial_\mu c$. Thus we have introduced external fields coupled to a system of operators which is closed under Slavnov transformations. The most general lagrangian invariant under charge conjugation , $\phi\pi$ neutral and consistent with power counting is now

$$\Sigma(\varphi, J, \eta, \gamma, \vec{\beta}, \tau) = \mathcal{L}(\varphi) + \eta_1(z_1 \bar{c}\varphi_1 + z_1' \bar{c}) + \eta_2 z_2 \bar{c}\varphi_2$$

$$+\gamma(a_1 \varphi_1\varphi_2 + a_2 \varphi_2 + a_3 \partial_\mu A_\mu) + \beta_i(B_{ij}\bar{\varphi}_j + M_i \varphi_i \bar{c}$$

$$+ v_i \bar{c} + w_i \Box \bar{c}) + \tau a_4 A_\mu \bar{c} \partial_\mu \bar{c} + a_5 \gamma^2 + J_1 \varphi_1 \qquad \text{(AVII,5)}$$

$$+ J_2 \varphi_2 + J_\mu A_\mu + \bar{\xi} c + \xi \bar{c}$$

and the Slavnov identity assumes the general form which will be needed later:

$$\delta_\sigma \Sigma^e = \delta \Sigma^e + \int dx \left[-\bar{\sigma}\bar{\xi}\delta_\gamma + \gamma\{\lambda_1 \Box \delta_\xi \right. \qquad \text{(AVII,6)}$$

$$\left. + \lambda_2 \delta_\xi + \lambda_3 \delta_{\eta_1} + \vec{b}\,\delta_{\vec{\beta}}\} + \vec{d}\cdot\vec{\beta}\,\delta_\tau \right](x)\, \Sigma^e = 0$$

where however in spite of the huge number of parameters involved in (AVII,5) the coefficients in δ_σ are always constrained by

$$\vec{d}\cdot\vec{b} = 0 \qquad \text{(AVII 7)}$$

because $b_i \propto (B^{-1})_{1i} - (B^{-1})_{2i}$ and $d_i \propto B_{i3}$ $(i=1,2,3)$.

One can in fact verify directly that this is the only constraint on the coefficients of the Slavnov identity .

We are now ready to prove that one can fulfill a Slavnov identity of the type (AVII,6) to all orders in $\hbar$, with coefficients constrained by (AVII,7)

Performing a Slavnov variation of the type (AVII,1) on an effective lagrangian

of the form * $N_4 \mathcal{L}(\varphi, I, \eta, \gamma, \vec{\beta}, \tau)$ (cf Eq.(AVII,5)) yields:

$$\delta_\sigma Z^e = \sum_1^{32} {}' c_i \Delta_i' Z^e \qquad \text{(AVII,8)}$$

where the first 23 Δ_i's are listed in Eq.(26), and the last nine Δ_i's are:

$$(24)\ \int dx\, [\gamma \bar{c}](x)$$
$$(25)\ \int dx\, [\gamma \Box \bar{c}](x)$$
$$(26)\ \int dx\, [\gamma N_3[\bar{c}\varphi_1]](x)$$
$$(27)\ \int dx\, [\gamma N_3[\bar{c}\varphi_1^2]](x)$$
$$(28)\ \int dx\, [\gamma N_3[\bar{c}\varphi_2^2]](x)$$
$$(29)\ \int dx\, [\gamma N_3[\bar{c} A_\mu A_\mu]](x)$$
$$(30,31,32)\ \int dx\, [\beta_i N_4[A_\mu \bar{c}\partial_\mu \bar{c}]](x) \qquad \text{(AVII,9)}$$

After elimination of the Δ_i's which are naïve variations, one is left with an $\mathcal{L}^{\text{eff}}$, including source terms such that only Δ_{21}, Δ_{22}, Δ_{23} and a linear combination of Δ_{30}, Δ_{31}, Δ_{32}, namely : $\Delta_{33} = \vec{b} \int dx [\vec{\beta} N_4[A_\mu \bar{c}\partial_\mu \bar{c}]](x)$ remain on the right hand side of the Slavnov identity which reads :

$$\mathcal{S} Z^e = (c_{21}\Delta_{22} + c_{22}\Delta_{22} + c_{23}\Delta_{23} + c_{33}\Delta_{33}) Z^e \qquad \text{(AVII,10)}$$

Now recall that $\mathcal{S}$ is the naïve Slavnov identity associated with the Slavnov transformation we started with, hence $\vec{d}\cdot\vec{b} = 0$

Now compute $\mathcal{S}^2 Z^e$, which because of this condition has the same form as the $\Phi\Pi$ equation of motion integrated through $\bar{\xi}$.

We have:

$$\mathcal{S}^2 Z^e = \mathcal{S}[c_{21}\Delta_{21} + c_{22}\Delta_{22} + c_{23}\Delta_{23} + c_{33}\Delta_{33}] Z^e \qquad \text{(AVII,11)}$$

The same argument as before shows that $\mathcal{S}^2 Z^e$ has coefficients identical with those occuring in the $\Phi\Pi$ equation of motion, except for the mass term, and that

$$c_{21} = c_{22} = c_{23} = c_{33} = 0 \qquad \text{(AVII,12)}$$

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* If a monomial is of the form $\varepsilon M(\varphi)$ where ε is an external field to which dimension d was assigned, $N_4[\varepsilon M(\varphi)]$ means $\varepsilon N_{4-d}[M(\varphi)]$.

At this point the Lagrangian depends on 24 parameters since 28 relations were imposed on the initial 52 parameters. Together with the 14 independant parameters of the Slavnov identity we have 38 parameters which can be fixed by the 9 normalization conditions in Eq.(31) [x] together with 26 others fixing the couplings with the external fields [xx]. It is a matter of routine to verify that the corresponding system is soluble the condition $\vec{a}\cdot\vec{b}=0$ being preserved Three parameters are then left free : e_2, a, σ. The gauge parameter σ could be fixed by imposing an extra normalization condition on $\Gamma_{\varphi_1^2\varphi_2^2}$
We now extend the proof of the gauge invariance of the Scattering operator. In order to do so, we shall decompose again the insertion Δ_λ generating an infinitesimal variation of the gauge parameter λ according to

$$\Delta_\lambda = \hat{\Delta}_\lambda + \Delta^S_\lambda \qquad \text{(AVII,13)}$$

and we shall show that it is possible to choose the two insertions $\hat{\Delta}_\lambda$ and Δ^S_λ satisfying the same requirements as in chapter III. (Eq.(68) and Eq.(73))
First of all, let us write Eq.(AVII,6) in the form.

$$\mathcal{S}_\sigma \mathcal{Z}^e \equiv \left\{ \sum_0^9 i\, c_i \mathcal{S}_i + \int dx \left[\gamma \vec{b} \delta_{\vec{\beta}} + \vec{a}\cdot\vec{\beta}\, \delta_\tau \right](x) \right\} \mathcal{Z}^e \qquad \text{(AVII,14)}$$

the first six $\mathcal{S}_i$'s are listed in Eq.(76) the remaining four are:

$$\begin{aligned}
\mathcal{S}_6 &= \int dx \left[\bar{\xi} \delta_\gamma \right](x) \\
\mathcal{S}_7 &= \int dx \left[\gamma \Box \delta_{\bar{\xi}} \right](x) \\
\mathcal{S}_8 &= \int dx \left[\gamma \delta_\xi \right](x) \qquad \text{(AVII,15)} \\
\mathcal{S}_9 &= \int dx \left[\gamma \delta_\eta \right](x)
\end{aligned}$$

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x Using the same kind of arguments as in Appendix III it can be shown that the condition given in Eq.(31,8) is a suitable normalization condition and that the mass term in the $\phi\pi$ equation of motion has the same coefficient as the corresponding term in $\mathcal{S}\mathcal{Z}^e$.

xx The simplest additional normalization conditions are:

$$z_1 = z_2 = a = B_{11} = B_{22} = B_{33} = a_4 = 1$$
$$z'_1 = a_2 = a_3 = a_5 = \vec{\mu} = \vec{\nu} = \vec{w} = B_{i\neq j} = 0$$

The derivative of δ_σ with respect to the parameter λ is obtained by differenciating the c_i's and the vectors $\vec{b}$ and $\vec{d}$. Since we know that $\vec{b}.\vec{d}=0$ (Eq.(AVII,7)) independently of λ, we have the equation :

$$\vec{d}\cdot\partial_\lambda\vec{b} + \vec{b}\,\partial_\lambda\vec{d} = 0 \qquad \text{(AVII,16)}$$

We can parametrize $\partial_\lambda\vec{b}$ and $\partial_\lambda\vec{d}$ by introducing the two cartesian triplets:

$$\vec{b},\ \vec{b}_1,\ \vec{b}_2 \qquad \text{(AVII,17)}$$
$$\vec{d},\ \vec{d}_1,\ \vec{d}_2$$

in the form:

$$\partial_\lambda\vec{b} = \sum_{i=1,2} x_i^{(\lambda)}\vec{d}_i + z^{(\lambda)}b^2\vec{d}$$
$$\partial_\lambda\vec{d} = \sum_{i=1,2} y_i^{(\lambda)}\vec{b}_i - z^{(\lambda)}d^2\vec{b} \qquad \text{(AVII,18)}$$

Thus we have to find a non physical $\hat{\Delta}_\lambda$ satisfying the equation:

$$i\hbar[\hat{\Delta}_\lambda, \mathcal{S}_\sigma] = \sum_i \partial_\lambda c_i^{(\lambda)}\vartheta_i + \int dx \Big\{\sum_{i=1,2} x_i^{(\lambda)}\gamma\vec{d}_i\,\delta_{\vec{\beta}}$$
$$+\sum_{i=1,2} y_i^{(\lambda)}\vec{b}_i\cdot\vec{\beta}\,\delta_\tau + z^{(\lambda)}(b^2\gamma\vec{d}\cdot\delta_{\vec{\beta}} - d^2\vec{b}\cdot\vec{\beta}\,\delta_\tau)\Big\}(x) \qquad \text{(AVII,19)}$$

The insertion $\hat{\Delta}_\lambda$ is a linear combination of a basis of covariant non physical insertions which can be found as follows.

First we introduce, in analogy with Eq.(78) three operators $\mathcal{Q}_{i,\varepsilon}$ with $\delta_{\mathcal{Q}\varepsilon} = \delta_\gamma$. By the same construction as in Chapter III we get three insertions Δ_i^0 of covariances $c_i\vartheta_i$ (i=4,5,6).

Then using the generalized action principle[6] we can complete the basis of covariant non physical insertions as indicated in table (AVII,1)

Insertion	Covariance
$\Delta_{\eta_2} = \frac{\hbar}{i}\int dx\,[\eta_2\delta_{\eta_2}](x)$	$-c_1\vartheta_1$
$\Delta_{\eta_1} = \frac{\hbar}{i}\int dx\,[\eta_1\delta_{\eta_1}](x)$	$-c_2\vartheta_2 - c_9\vartheta_9$
$\Delta'_{\eta_1} = \frac{\hbar}{i}\int dx\,[\eta_1\delta_{\xi}](x)$	$-c_2\vartheta_3 - c_9\vartheta_8$
$\Delta_\beta = \frac{\hbar}{ib^2}\int dx\,[\vec{b}\cdot\vec{\beta}\,\Box\delta_\xi](x)$	$-\vartheta_7$
$\Delta'_\beta = \frac{\hbar}{ib^2}\int dx\,[\vec{b}\cdot\vec{\beta}\,\delta_\xi](x)$	$-\vartheta_8$
$\Delta''_\beta = \frac{\hbar}{ib^2}\int dx\,[\vec{b}\cdot\vec{\beta}\,\delta_{\eta_1}](x)$	$-\vartheta_9$
$\Delta^i_{\beta^2} = \frac{\hbar}{ib^2}\int dx\,[\vec{b}\cdot\vec{\beta}\,\vec{d}_i\,\delta_{\vec{\beta}}](x)$	$-\int dx\,[\gamma\vec{d}_i\delta_{\vec{\beta}}](x)$
$\Delta''^i_{\beta^2} = \frac{\hbar}{id^2}\int dx\,[\vec{b}_i\cdot\vec{\beta}\,\vec{d}\,\delta_{\vec{\beta}}](x)$	$\int dx\,[\vec{b}_i\cdot\vec{\beta}\,\delta_\tau](x)$
$\Delta'^i_{\beta^2} = \frac{\hbar}{i}\int dx\,[\vec{b}\cdot\vec{\beta}\,\vec{d}\cdot\delta_{\vec{\beta}}](x)$	$\int dx\,[d^2\vec{b}\cdot\vec{\beta}\,\delta_\tau - b^2\gamma\vec{d}\cdot\delta_{\vec{\beta}}](x)$

TABLE AVII-I-

It is evident that one can find a particular solution of Eq.(AVII,19) as a linear combination of these insertions and of the three
In the same way one can get other non physical insertions which are listed in table (AVII-II)

Insertions	Covariance
$\Delta^0_i = \int dx\, J_i(x)$	0
$\Delta_c = \frac{\hbar}{i}\int dx\,[\xi\,\delta_{\xi}](x)$	$c_4\vartheta_4 + c_5\vartheta_5 + c_6\vartheta_6$
$\Delta_{\varphi_2} = \frac{\hbar}{i}\int dx\,[J_2\,\delta_{J_2}](x)$	$c_2\vartheta_2 + c_3\vartheta_3 - c_5\vartheta_5$
$\Delta_\gamma = \frac{\hbar}{i}\int dx\,[\gamma\,\delta_\gamma](x)$	$-c_6\vartheta_6 + \sum_{i=7,8,9} c_i\vartheta_i + \gamma\vec{b}\cdot\delta_{\vec{\beta}}$
$\Delta'_\gamma = \frac{\hbar}{i}\int dx\,[\gamma\,\delta_{J_2}](x)$	$c_3\vartheta_8 + c_2\vartheta_9 - c_6\vartheta_5$
$\Delta''_\gamma = \frac{\hbar}{i}\int dx\,[\gamma\,\partial_\mu\,\delta_{J_\mu}](x)$	$\vartheta_7 - c_6\vartheta_4$
$\Delta^i_\beta = \frac{\hbar}{i}\int dx\,[\vec{b}_i\cdot\delta_{\vec{\beta}}\,\Box\,\delta_\xi](x)$	0
$\Delta'^i_\beta = \frac{\hbar}{i}\int dx\,[\vec{b}_i\cdot\delta_{\vec{\beta}}\,\delta_\xi](x)$	0
$\Delta''^i_\beta = \frac{\hbar}{i}\int dx\,[\vec{b}_i\cdot\delta_{\vec{\beta}}\,\delta_{\eta_i}](x)$	0
$\Delta^{ij}_{\beta^2} = \frac{\hbar}{i}\int dx\,[\vec{b}_i\cdot\vec{\beta}\;\vec{d}_j\cdot\delta_{\vec{\beta}}](x)$	0
$\Delta_\tau = \frac{\hbar}{i}\int dx\,[\tau\,\delta_\tau](x)$	$-\int dx\,[\vartheta_\beta\,\delta_c](x)$
$\Delta_{\gamma^2} = \int dx\,\gamma^2(x)$	$= \bar{\sigma}\int dx\,[\xi\,\gamma](x) = -\frac{\hbar\bar{\sigma}}{i\sqrt{2}}\int dx\,[\vartheta(\bar{m}\delta^i_\xi)](x)$

TABLE AVII -II-

It is clear that combining linearly the insertions listed in Table(AVII.II) with those previously considered we obtain 18 symmetrical non physical insertions; in fact, because of the orthogonality condition Eq.(AVII,7) $\vec{b}$ is a linear combination of the $\vec{d}_i$'s and $\vec{d}$ of the $\vec{b}_i$'s. (i=1,2).

Now, following the same procedure as in Chapter III, we complete the construction of Δ_λ by studying a basis of symmetrical insertions Δ^S.

Since we know that, given the Slavnov identity, the complete lagrangian (Eq.(AVII,5)) depends on 24 parameters, (6 of them fixing the propagators and the couplings of the quantized fields, and 18 specifying the external field dependence) it follows that there are 24 independent symmetrical insertions. We have already constructed 18 independent Δ^S's which are non physical. Thus to complete the proof of gauge invariance we have to find six symmetrical insertions satisfying Eq.(70). Five of them are determined by the independent terms of the tree approximation lagrangian (Eq.(AVII,2) excluding $\frac{g^2}{2} + \bar{c}Mc$. The sixth one is the analog of Δ^S_6 (Eq.(98)). They verify Eq.(70) as can be seen in the tree approximation.

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